The Wigner rotation for photons in an arbitrary gravitational field

P.M. Alsing†*
*Air Force Research Laboratory, Space Vehicles Directorate
3550 Aberdeen Ave, SE, Kirtland AFB, New Mexico, 87117-5776

G.J. Stephenson Jr†
†Department of Physics and Astronomy, University of New Mexico, Albuquerque, NM 87131
(Dated: February 10, 2009 Version: v3)

We investigate the Wigner rotation for photons, which governs the change in the polarization of the photon as it propagates through an arbitrary gravitational field. We give explicit examples in Schwarzschild spacetime, and compare with the corresponding flat spacetime results, which by the equivalence principle, holds locally at each spacetime point. We discuss the implications of the Wigner rotation for entangled photon states in curved spacetime, and lastly develop a sufficient condition for special (Fermi-Walker) frames in which the observer would detect no Wigner rotation.

I. INTRODUCTION

Recently, there has been much interest in the study of entanglement for moving observers, both for constant velocity observers (special relativity - SR) and for arbitrarily accelerated observers (general relativity - GR). An excellent, recent review can be found in Peres and Terno [1] (and references therein). In SR and GR the important ingredient that determines the description of moving states by moving observers is how such states transform under the symmetries that govern the underlying flat or curved spacetime. The relevant concept is that of the *Wigner rotation* [2], which for massive particles, mixes up the spin components (along a given quantization axis) for a particle of definite spin by an $O(3)$ rotation, and for massless particles, introduces a phase factor which is the product of a Wigner rotation angle times the helicity of the state. In this paper we investigate the transformation of photon states as they traverse trajectories in an arbitrary curved spacetime (CST), and investigate the implications for the evolution of entangled states. In a companion article [3], similar investigations were carried out on the role of the Wigner rotation on the entanglement of massive spin $\frac{1}{2}$ particles in CST.

This paper is organized as follows. In Section II we review the transformation of quantum mechanical states of massive particles under Lorentz transformations, in both flat and curved spacetime, and the consequences for entangled states. In Section III we review the current research into similar investigations for photons in flat spacetime. In Section IV we generalize the flat spacetime results of the previous section to CST and give specific examples of the Wigner rotation in the spherically symmetric Schwarzschild spacetime. In Section V we derive sufficient conditions for the existence of reference frames for observers to measure a null Wigner rotation. In Section VI we consider the consequences of the Wigner rotation on the entanglement of photon states and photon wavepackets in CST. In Section VII we present a summary and our conclusions. In the appendix we review the effect of the Wigner rotation on the rotation of the photon polarization in the plane perpendicular to its propagation direction in flat spacetime. By Einstien's equivalence principle (EP), which states that SR applies in the locally flat (Lorentz) tangent plane to a point x in CST, the flat spacetime examples presented have relevance when the Wigner rotation is generalized to photons and arbitrarily moving observers in CST.

II. MASSIVE PARTICLES WITH SPIN

A. Flat Spacetime

In quantum field theory, the merger of quantum mechanics with SR, the particle states for massive particles are defined by their spin (as in non-relativistic mechanics) in the particle's rest frame, and additionally by their momentum. These two quantities are the Casimir invariants of the ten parameter Poincare group of SR which describes ordinary rotations, boosts and translations. The positive energy, single particle states thus form the state space for a representation of the Poincare group. For massive particles with spin j, the states are given by $|\vec{p},\sigma\rangle$, where $\vec{p}$ is the spatial portion of the particle's 4-momentum p^μ, and $\sigma \in \{-j,-j+1,\ldots,j\}$ are the components of the particle's spin along a quantization axis in the rest frame of the particle. For massless particles, the states are given by $|\vec{p},\lambda\rangle$ where λ indicates the helicity states of the particle ($\lambda=\pm1$ for photons, $\lambda=\pm1/2$ for massless fermions).

Under a Lorentz transformation (LT) $\mathbf{\Lambda}$ the single-particle state for a massive particle transforms under the unitary transformation $U(\Lambda)$ as [2]

$$U(\Lambda)|\vec{p},\sigma\rangle = \sum_{\sigma'} D^{(j)}_{\sigma'\sigma}(W(\Lambda,\vec{p}))\,|\overrightarrow{\Lambda p},\sigma'\rangle, \qquad (1)$$

*Electronic address: paul.alsing@kirtland.af.mil; Corresponding author
†Electronic address: gjs@swcp.com

where $\overrightarrow{\Lambda p}$ are the spatial components of the Lorentz transformed 4-momentum, i.e. $\vec{p}^{\,\prime}$ where $p^{'\mu} = \Lambda^{\mu}{}_{\nu}\, p^{\nu}$. In Eq.(1), $D^{j}_{\sigma'\sigma}(W(\Lambda,\vec{p}))$ is a $(2j+1)\times(2j+1)$ matrix spinor representation of the rotation group $O(3)$, and $W(\Lambda,\vec{p})$ is called the Wigner rotation angle. The explicit form of the Wigner rotation in matrix form is given by

$$\boldsymbol{W}(\Lambda,\vec{p}) = \boldsymbol{L}^{-1}(\Lambda p)\cdot\boldsymbol{\Lambda}\cdot\boldsymbol{L}(p), \tag{2}$$

where $\boldsymbol{L}(p)$ is a *standard boost* taking the standard rest frame 4-momentum $\boldsymbol{k} \equiv (m,0,0,0)$ to an arbitrary 4-momentum $\boldsymbol{p}$, $\boldsymbol{\Lambda}$ is an arbitrary LT taking $\boldsymbol{p}\to\boldsymbol{\Lambda}\cdot\boldsymbol{p}\equiv\boldsymbol{\Lambda p}$, and $\boldsymbol{L}^{-1}(\Lambda p)$ is an inverse standard boost taking the final 4-momentum $\boldsymbol{\Lambda p}$ back to the particle's rest frame. Because of the form of the standard rest 4-momentum $\boldsymbol{k}$, this final rest momentum $\boldsymbol{k}^{'}$ can at most be a spatial rotation of the initial standard 4-momentum $\boldsymbol{k}$, i.e. $\boldsymbol{k}^{'} = \boldsymbol{W}(\Lambda,\vec{p})\cdot\boldsymbol{k}$. The rotation group $O(3)$ is then said to form (Wigner's) *little group* for massive particles, i.e. the invariance group of the particle's rest 4-momentum. The explicit form of the standard boost is given by [2]

$$\begin{aligned}
L^{0}{}_{0} &= \gamma = \frac{p^{0}}{m} \\
L^{i}{}_{0} &= \frac{p^{i}}{m}, \quad L^{0}{}_{i} = -\frac{p_{i}}{m}, \\
L^{i}{}_{j} &= \delta^{i}{}_{j} - (\gamma-1)\,\frac{p^{i}p_{j}}{|\vec{p}|^{2}}, \qquad i,j=(1,2,3),
\end{aligned} \tag{3}$$

where $\gamma = p^{0}/m = E/m \equiv e$ is the particles energy per unit rest mass. Note that for the flat spacetime metric $\eta_{\alpha\beta}=$diag$(1,-1,-1,-1)$, $p_{0}=p^{0}$ and $p_{i}=-p^{i}$.

Peres, Scudo and Terno [4] considered a free spin 1/2 particle in a normalizable state containing a distribution of momentum states. Under a LT, Eq.(1) indicates that each component of the momentum will undergo a different Wigner rotation, since the later is momentum dependent. Therefore, the reduced spin density matrix for the particle, obtained by tracing over the momentum states, will have a non-zero von Neumann entropy, which will increase as with the rapidity r of the observer with constant velocity v (with $\tanh r = v/c$). This indicates that the spin entropy of the particle is not a relativistic scalar and thus has no invariant meaning.

Alsing and Milburn [5] considered the transformation of Bell states of the form

$$\begin{aligned}
|\Psi^{\pm}\rangle &= [\,|\vec{p},\uparrow\rangle\,|-\vec{p},\uparrow\rangle \pm |\vec{p},\downarrow\rangle\,|-\vec{p},\downarrow\rangle\,]/\sqrt{2}, \\
|\Phi^{\pm}\rangle &= [\,|\vec{p},\uparrow\rangle\,|-\vec{p},\downarrow\rangle \pm |\vec{p},\downarrow\rangle\,|-\vec{p},\uparrow\rangle\,]/\sqrt{2},
\end{aligned} \tag{4}$$

composed of pure momentum eigenstates. Under a Lorentz boost perpendicular to the motion of the particle, the transformed momentum $\vec{p}^{\,\prime}$ is rotated by an angle θ with respect to the original momentum $\vec{p}$, while the direction of the particle's spin is rotated slightly less by the momentum dependent Wigner angle $\Omega_{p}<\theta$. The implication is that if the boosted observer were to orient his detectors along $\pm\vec{p}^{\,\prime}$ at angle $\pm\theta$ there could be an apparent degradation of the violation of Bell inequalities in that inertial frame. However, if the observer were to orient his detectors along the direction of the transformed spins $\pm\Omega_{p}$, there would again be a maximal violation of the Bell inequalities for these states in the boosted inertial frame. The entanglement of the complete bipartite state is preserved.

Gingrich and Adami [6] considered a normalizable Bell state in which (analogous to Peres, Scudo and Terno) there is a distribution over momentum states. Considering a Bell state $|\Phi^{+}\rangle$ with the momentum in a product Gaussian distribution, a LT will transfer the spin entanglement into the momentum. If one forms the reduced 2-qubit spin density matrix, it will exhibit a Wootter's concurrence [7] which decreases with increasing rapidity. Because the quantum states contain two degrees of freedom, spin and momentum, the LT induces a *spin-momentum entanglement* again due to the momentum dependent Wigner rotation Eq.(1). Similar investigations were carried out for the case of photons [5, 8, 9], which we shall return to shortly.

B. Curved Spacetime

Subsequently, Terashima and Ueda [10] extended the definition of the Wigner rotation of spin 1/2 particles to an arbitrary gravitational field. The essential point in going from SR to GR is that there are in general no global inertial frames, only *local frames*. In fact, in GR all reference frames are allowed, locally inertial (i.e. zero acceleration trajectories - geodesics) or otherwise. Einstein's Equivalence Principle (EP) states that in an arbitrary curved spacetime, SR holds locally in the tangent plane to a given event in spacetime at the point x. The positive energy single particle states $|\vec{p}(x),\sigma\rangle$ now form the state space for a *local* representation of the Poincare group in the locally flat Lorentz tangent plane at each spacetime point x.

Of particular importance is the observer's local reference frame which can be described by set of four axes (4-vectors) called a tetrad $\mathbf{e}_{\hat{a}}(x)$ with $\hat{a}\in\{0,1,2,3\}$ [11]. Three of these axes $\hat{i}\in\{1,2,3\}$ describe the spatial axes at the origin of the observer's local laboratory, from which he makes local measurements. The fourth axis $\hat{a}=0$ describes the rate at which a clock, carried by the observer at the origin of his local laboratory, ticks (gravitational redshift effect), and is taken to be the observer's 4-velocity $\mathbf{e}_{\hat{0}}(x)\equiv\mathbf{u}$, where $\mathbf{u}$ is the tangent to the observer's worldline through spacetime.

In curved spacetime (CST) described by coordinates x^{α}, we can define *coordinate basis vectors* (CBV) $\boldsymbol{e}_{\mu}(x)$ with components $e_{\mu}^{\ \alpha}(x)=\delta_{\mu}^{\ \alpha}$. We interpret the CBV $\boldsymbol{e}_{\mu}(x)$ as a local vector at the spacetime point x, pointing along the coordinate direction x^{μ}. These are not unit vectors since their inner product yields the spacetime metric, $\boldsymbol{e}_{\mu}(x)\cdot\boldsymbol{e}_{\mu}(x)\equiv g_{\alpha\beta}(x)\,e_{\mu}^{\ \alpha}(x)\,e_{\nu}^{\ \beta}(x) = g_{\mu\nu}(x)$. The observer's tetrad $\{\mathbf{e}_{\hat{a}}(x)\}$ form an *or-*

thonormal basis (ONB, denoted by carets over the indices) such that the inner product of any two basis vectors forms the flat spacetime metric of special relativity, $\mathbf{e}_{\hat{a}}(x)\cdot\mathbf{e}_{\hat{b}}(x)\equiv g_{\alpha\beta}(x)\,e_{\hat{a}}{}^{\alpha}(x)\,e_{\hat{b}}{}^{\beta}(x)=\eta_{\hat{a}\hat{b}}$ where $\eta_{\hat{a}\hat{b}}\equiv$ diagonal$(1,-1,-1,-1)$ is the metric of SR. Note that $e_{\hat{a}}{}^{\alpha}(x)$ are the components of the ONB vectors written in terms of the CBVs, $\mathbf{e}_{\hat{a}}(x)=e_{\hat{a}}{}^{\alpha}(x)\,\mathbf{e}_{\alpha}(x)$. That the metric $g_{\alpha\beta}(x)$ can be brought into the form $\eta_{\hat{a}\hat{b}}$ *locally* at each spacetime point x (since the tetrad components $e_{\hat{a}}{}^{\alpha}(x)$ are spacetime dependent) is an embodiment of Einstein's Equivalence Principle (EP) which states that SR holds locally at each spacetime point x in an arbitrary curved spacetime (CST). In Section IV B we will exhibit explicit examples of tetrads for various types of observers.

Tensor quantities $T^{\gamma\delta}_{\alpha\beta}(x)$ which "live" in the surrounding spacetime described by coordinates x^{α}, (called *world* tensors, and denoted by Greek indices), transform according to general coordinate transformations (GCT) $x^{\alpha}\rightarrow x'^{\alpha}(x)$, which simply describe the same spacetime in the new coordinates. World tensors can be projected down to the observer's local frame via the components of the tetrad and its inverse, i.e. $T^{\hat{c}\hat{d}}_{\hat{a}\hat{b}}(x)=e_{\hat{a}}{}^{\alpha}(x)\,e_{\hat{b}}{}^{\beta}(x)\,e_{\gamma}{}^{\hat{c}}(x)\,e_{\delta}{}^{\hat{d}}(x)\,T^{\gamma\delta}_{\alpha\beta}(x)$. Here, $e^{\hat{a}}{}_{\mu}(x)$ is the inverse transpose of the matrix of tetrad vectors $e_{\hat{a}}{}^{\mu}(x)$, satisfying $\mathbf{e}^{\hat{a}}(x)\cdot\mathbf{e}^{\hat{b}}(x)\equiv g^{\alpha\beta}(x)\,e^{\hat{a}}{}_{\alpha}(x)\,e^{\hat{b}}{}_{\beta}(x)=\eta^{\hat{a}\hat{b}}$, where $g^{\alpha\beta}(x)$ and $\eta^{\hat{a}\hat{b}}$ are the inverses of the CST and flat spacetime metrics $g_{\alpha\beta}(x)$ and $\eta_{\hat{a}\hat{b}}$, respectively. We denote the observer's local components $T^{\hat{c}\hat{d}}_{\hat{a}\hat{b}}(x)$ of the world tensor $T^{\gamma\delta}_{\alpha\beta}(x)$ with hatted Latin indices. Objects with hatted indices transform as scalars with respect to GCTs, but as *local Lorentz vectors* with respect to local Loretnz transformations (LLT) $\Lambda(x)$, which simply transform between different instantaneous local Lorentz frames, or instantaneous states of motion of different types of observers, at the point x (e.g. stationary, freely falling, circular orbit, etc. . .).

In particular, the local components $p^{\hat{a}}(x)$ of the world 4-momentum $p^{\alpha}(x)=m\,u^{\alpha}(x)$ of a particle (with mass m and 4-velocity $u^{\alpha}(x)$) passing through the observer's local laboratory at the spacetime point x are given by $p^{\hat{a}}(x)=e^{\hat{a}}{}_{\alpha}(x)\,p^{\alpha}(x)=\mathbf{e}^{\hat{a}}(x)\cdot\mathbf{p}(x)$. In SR, $\mathbf{p}_{SR}=(E/c,\vec{p})$ is the 4-momentum of the particle, whose time component is the particle's energy and whose spatial components are its 3-momentum. In CST, $p^{\hat{0}}(x)$ is energy of the particle with 4-momentum $\mathbf{p}$ as measured by the observer with tetrad $\{\mathbf{e}_{\hat{a}}(x)\}$ as the particle passes through his local laboratory at the spacetime point x, while $p^{\hat{i}}(x)$ are the locally measured components of the 3-momentum. Since GR allows for observer undergoing arbitrary motion (as opposed to SR which considers only zero acceleration or constant velocity observers, i.e. inertial frames), the observer's locally measured components of the 4-momentum $p^{\hat{a}}(x)$ depends upon the observer's state of motion, described by the motion of the axes comprising the his local laboratory, which are given by his tetrad $\{\mathbf{e}_{\hat{a}}(x)\}$. As a field of vectors over the spacetime, we interpret the the tetrad $\{\mathbf{e}_{\hat{a}}(x)\}$ as a collection of observer's located at each spacetime point x, usually of a particular type (stationary, freely falling, circular orbit, etc. . .). We shall see explicit examples of different types of observers (tetrads) in Section IV B.

As the passing (massive) particle moves in an arbitrary fashion from $x^{\alpha}\rightarrow x^{\alpha'}=x^{\alpha}+u^{\alpha}(x)\,d\tau$ in infinitesimal proper time $d\tau$, Terashima and Ueda found that the local momentum components $p^{\hat{a}}(x)$ would change under an infinitesimal LT, $\lambda^{\hat{a}}{}_{\hat{b}}(x)$ via $p^{\hat{a}}(x)\rightarrow p^{\hat{a}'}(x)=p^{\hat{a}}(x)+\delta p^{\hat{a}}(x)$ where $\delta p^{\hat{a}}(x)=\lambda^{\hat{a}}{}_{\hat{b}}(x)\,p^{\hat{b}}(x)$ and $\Lambda^{\hat{a}}{}_{\hat{b}}(x)=\delta^{\hat{a}}{}_{\hat{b}}(x)+\lambda^{\hat{a}}{}_{\hat{b}}(x)\,d\tau$ is the local Lorentz transformation (LLT) to first order in $d\tau$. A straight forward, though lengthy calculation (detailed in Alsing *et al* [3]), leads from $\lambda^{\hat{a}}{}_{\hat{b}}(x)$ to the infinitesimal Wigner rotation $\vartheta^{\hat{a}}{}_{\hat{b}}(x)$ where $W^{\hat{a}}{}_{\hat{b}}(x)=\delta^{\hat{a}}{}_{\hat{b}}(x)+\vartheta^{\hat{a}}{}_{\hat{b}}(x)\,d\tau$ is the local Wigner rotation to first order in $d\tau$. Terashima and Ueda showed that only the space-space components of $\vartheta^{\hat{a}}{}_{\hat{b}}(x)$ are nonzero, and thus it truly represents an infinitesimal $\mathcal{O}(3)$ rotation for massive particles.

In the instantaneous non-rotating rest frame of the particle for geodesic motion (zero local acceleration), Alsing *et al* [3] showed that the Wigner rotation would be zero. Further, considering the $\mathcal{O}(\hbar)$ quantum correction to the non-geodesic motion of spin 1/2 particles in an arbitrary gravitational field the authors showed that in the instantaneous non-rotating rest frame of the accelerating particle (the Ferimi-Walker (FW) frame) the Wigner rotation would also be observed to be zero.

III. PHOTONS

A. Flat Spacetime

In the above, we have primarily considered massive spin 1/2 particles. As briefly discussed above, analogous results have been obtained for photons in SR [5, 8, 9]. The fundamental difference between going from massive to massless particles is there is no rest frame for the latter. Instead, one defines a *standard frame* in which the photon 4-momentum takes the form $\tilde{k}^{\mu}=(1,0,0,1)$ where the photon travels in a predefined direction along the $\mathbf{z}$-axis, and LT this to a photon of 4-momentum $\mathbf{k}$ of arbitrary energy k^0 and traveling in an arbitrary direction, subject to the null condition $\mathbf{k}\cdot\mathbf{k}\equiv k^{\mu}\,k_{\mu}=0$. The photon 4-momentum has the form $\mathbf{k}\leftrightarrow k^{\mu}=(k^0,|\vec{k}|\,\hat{\boldsymbol{k}})$ where $\vec{k}$ is the spatial 3-momentum of the photon, $\hat{\boldsymbol{k}}=\vec{k}/|\vec{k}|=\vec{n}$ is the direction of propagation of the photon and $\omega\equiv k^0=|\vec{k}|$ is the frequency of the photon. Consequently, Wigner's little group for massless particles is $SO(2)$, the group of rotations and translations in two dimensions associated with the particle's transverse plane of polarization. Under a LT Λ taking $\mathbf{k}\rightarrow\mathbf{k}'=\Lambda\,\mathbf{k}$ the

transformation of the photon helicity state $|\mathbf{k}, \lambda\rangle$, analogous to Eq.(1) is given by [8, 9]

$$U(\Lambda)\,|\mathbf{k}, \lambda\rangle = e^{i\,\lambda\,\psi(\Lambda,k)}|\Lambda\mathbf{k}, \lambda\rangle, \qquad (5)$$

where $\psi(\Lambda, k)$ is the momentum-dependent Wigner rotation angle (phase). Due to the fact that $\mathbf{k}$ is a null 4-vector, The Wigner angle depends only on the direction of propagation of the photon $\hat{\boldsymbol{k}}$, and not on its frequency ω, i.e. $\psi(\Lambda, k) = \psi(\Lambda, \vec{n})$. Note that the unitary transformation in Eq.(5) does not change the helicity of the photon state, in contrast to the case for massive particles Eq.(1) in which the components of the spin are mixed up by a momentum dependent Wigner rotation.

The corresponding transformation of the polarization vectors for positive and negative helicity states $\epsilon^{\mu}_{\pm}$

$$\epsilon^{\mu}_{\pm}(\hat{\boldsymbol{k}}) = \frac{R(\hat{\boldsymbol{k}})}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ \mp i \\ 0 \end{bmatrix}, \qquad (6)$$

is given by [8]

$$\begin{aligned} \epsilon^{'\mu}_{\pm}(\hat{\boldsymbol{k}}') &\equiv D(\Lambda)\,\epsilon^{\mu}_{\pm}(\hat{\boldsymbol{k}}) \\ &= R(\Lambda\hat{\boldsymbol{k}})\,R_z(\psi(\Lambda,\vec{n}))\,R(\hat{\boldsymbol{k}})^{-1}\,\epsilon^{\mu}_{\pm}(\hat{\boldsymbol{k}}), \\ &= \Lambda\epsilon^{\mu}_{\pm}(\hat{\boldsymbol{k}}) - \frac{(\Lambda\,\epsilon^{\mu}_{\pm}(\hat{\boldsymbol{k}}))^0}{(\Lambda\,k^{\mu})^0}\,\Lambda k^{\mu}. \end{aligned} \qquad (7)$$

where $R(\Lambda\hat{\boldsymbol{k}})$ is the rotation taking the standard direction $\hat{\mathbf{z}}$ to $\hat{\boldsymbol{k}}$. Here we use the (abused) shorthand notation $\Lambda\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}' = \vec{k}'/|\vec{k}'|$. In the the appendix we illustrate several examples in flat spacetime of cases where the Wigner angle is zero, as well as cases in which it is non-zero, that will prove useful in our extension to curved spacetime below.

B. Curved Spacetime

In the following we extend the work of Terashima and Ueda [10] and Alsing *et al* [3] for massive spin 1/2 particles in curved spacetime to photons. Let $k_\mu(x)$ be the photon 4-momentum, and $e_{\hat{a}}{}^{\mu}(x)$ be the tetrad that defines the (timelike) observer's local laboratory. The local (covariant) components of the photon 4-momentum $k_{\hat{a}}(x)$, measured in the observer's laboratory, are given by projecting $k_\mu(x)$ onto the observer's local axes via the tetrad, $k_{\hat{a}}(x) = e_{\hat{a}}{}^{\mu}(x)\,k_\mu(x)$. We are interested in the change $\delta k_{\hat{a}}(x)$ of the locally measured photon components as the photon moves from $x^\mu \to x^{'\mu} = x^\mu + k^\mu(x)\,\delta\xi$ (where ξ is the affine parameter along the photon's trajectory). Following [3, 10] we compute

$$\delta k_{\hat{a}}(x) = \left(\delta e_{\hat{a}}{}^{\mu}(x)\right) k_\mu(x) + e_{\hat{a}}{}^{\mu}(x)\,\delta k_\mu(x). \qquad (8)$$

For the last term we have

$$\delta k_\mu(x) = d\xi\,\nabla_{\mathbf{k}}\,k_\mu(x) \equiv d\xi\,k^\beta(x)\nabla_\beta\,k_\mu(x) = 0, \qquad (9)$$

where ∇_β is the Riemann covariant derivative [11] constructed from the spacetime metric. The last equality is simply the definition that the photon's trajectory is a geodesic $\nabla_{\mathbf{k}}\mathbf{k} = 0$.

Using the orthonormality of the tetrad 4-vector axes $e_\nu{}^{\hat{b}}(x)\,e_{\hat{b}}{}^{\mu}(x) = \delta_\nu{}^{\mu}$, the first term in Eq.(8) becomes

$$\begin{aligned} \delta e_{\hat{a}}{}^{\mu}(x) &= d\xi\,\nabla_{\mathbf{k}}\,e_{\hat{a}}{}^{\mu}(x) = d\xi\,\left(\nabla_{\mathbf{k}}\,e_{\hat{a}}{}^{\nu}(x)\right) e_\nu{}^{\hat{b}}(x)\,e_{\hat{b}}{}^{\mu}(x), \\ &\equiv \chi_{\hat{a}}{}^{\hat{b}}(x)\,e_{\hat{b}}{}^{\mu}(x)\,d\xi, \end{aligned} \qquad (10)$$

where we have defined the local matrix $\chi_{\hat{a}}{}^{\hat{b}}(x)$ describing the rotation of the tetrad as

$$\chi_{\hat{a}}{}^{\hat{b}}(x) \equiv \left(\nabla_{\mathbf{k}}\,e_{\hat{a}}{}^{\nu}(x)\right) e_\nu{}^{\hat{b}}(x). \qquad (11)$$

Thus, the change in the local components $k_{\hat{a}}(x)$ of the photon's momentum as observed by the an observer with tetrad $e_{\hat{a}}{}^{\mu}(x)$ is given by the LLT

$$\begin{aligned} k_{\hat{a}}(x) &\to k'_{\hat{a}}(x) \equiv k_{\hat{a}}(x) + \delta k_{\hat{a}}(x) \\ &= \Lambda_{\hat{a}}{}^{\hat{b}}(x)\,k_{\hat{b}}(x) = \left(\delta_{\hat{a}}{}^{\hat{b}} + \lambda_{\hat{a}}{}^{\hat{b}}(x)\,d\xi\right)\,k_{\hat{b}}(x), \end{aligned} \qquad (12)$$

whose infinitesimal portion $\lambda_{\hat{a}}{}^{\hat{b}}(x)$ is given by

$$\delta k_{\hat{a}}(x) = \chi_{\hat{a}}{}^{\hat{b}}(x)\,k_{\hat{b}}(x)\,d\xi \equiv \lambda_{\hat{a}}{}^{\hat{b}}(x)\,k_{\hat{b}}(x)\,d\xi. \qquad (13)$$

Therefore, in CST the form of the transformation of a pure helicity photon state $|\mathbf{k}, \lambda\rangle$ has the same form as Eq.(5) if we interpret $\mathbf{k}$ as the the photon wavevector as measured by the observer described by the tetrad $\mathbf{e}_{\hat{a}}(x)$, i.e. with components $\mathbf{k} \leftrightarrow k^{\hat{a}}(x) = e^{\hat{a}}{}_{\alpha}(x)\,k^\alpha(x)$, and Λ a LLT as given in Eq.(12) for the infinitesimal motion of the photon along its geodesic from $x^\mu \to x^{'\mu} = x^\mu + k^\mu(x)\,\delta\xi$.

IV. WIGNER ROTATION ANGLE FOR PHOTONS IN CURVED SPACETIME

A. Derivation

Since in the instantaneous non-rotating rest frame of the observer the metric is locally flat $\eta_{\hat{a}\hat{b}} = (1, -1, -1, -1)$ we can follow the SR derivation of the Wigner phase $\psi(\Lambda, k)$ by Caban and Rembielinski [9], with the arbitrary LT in curved spacetime given by $\Lambda^{\hat{a}}{}_{\hat{b}}(x)$ in Eq.(12) to first order in $d\xi$. The authors' elegant derivation utilized the canonical homomorphism between $SL(2, C)$ and the proper orthochronos homogeneous Lorentz group $L^{\uparrow}_{+} \sim SO(1, 3)$. For every LLT $\Lambda(x) \in L^{\uparrow}_{+}$ of the photon 4-momentum $k^{\hat{a}}(x) \to k^{'\hat{a}}(x) \equiv \Lambda^{\hat{a}}{}_{\hat{b}}(x)\,k^{\hat{b}}(x)$ there corresponds a transformation of the Hermetian matrix $K(x) \equiv k^{\hat{a}}(x)\sigma_{\hat{a}}$ of the form $K(x) \to K'(x) = A_{k'}(x)K(x)A^{\dagger}_{k}(x)$ where $A_k(x) \in SL(2, C)$, $\sigma_{\hat{i}}$ are the constant Pauli matrices and $\sigma_{\hat{0}}$ is the 2×2 identity matrix. Henceforth we will drop the spacetime argument

x on all local quantities for notational clarity. For the most general photon 4-momentum $k^{\hat{a}} = (k^{\hat{0}}, k^{\hat{i}})$, K has the form

$$K = k^{\hat{0}} \begin{pmatrix} 1+n^{\hat{3}} & n_- \\ n_+ & 1-n^{\hat{3}} \end{pmatrix}, \quad (14)$$

where $n^{\hat{i}} = k^{\hat{i}}/k^{\hat{0}}$ and $n_\pm = n^{\hat{1}} \pm i\, n^{\hat{2}}$. The most general element $A \in SL(2,C)$ is of the form

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (15)$$

where the entries are complex numbers subject only to the normalization condition $\det A = \alpha\delta - \beta\gamma = 1$. We wish to relate these entries to the components of infinitesimal LLT $\lambda^{\hat{a}}{}_{\hat{b}}$, which is antisymmetric $\lambda_{\hat{b}\hat{a}} = -\lambda_{\hat{a}\hat{b}}$.

Corresponding to the expression for the Wigner rotation $W(\Lambda,k) = L^{-1}_{k'}\Lambda L_k$ of Eq.(1) with $p \to k$, is the $SL(2,C)$ transformation

$$S(\Lambda,k) = A^{-1}_{k'} A A_k. \quad (16)$$

Here A_k is the Lorentz boost that takes the standard photon 4-momentum $\tilde{k}^{\hat{a}} = (1,0,0,1)$ to $k^{\hat{a}} = (k^{\hat{0}}, k^{\hat{i}})$, A is an arbitrary LT Eq.(15), and $A^{-1}_{k'}$ is the inverse boost taking the transformed momentum $k'^{\hat{a}}$ back to the standard 4-momentum $\tilde{k}^{\hat{a}}$. The most general form of $S(\Lambda,k)$, the $SL(2,C)$ element of Wigner's little group that leaves $\tilde{K}$ invariant, is found by solving $\tilde{K} = S\tilde{K}S^\dagger_0$ yielding

$$S = \begin{pmatrix} e^{i\psi/2} & z \\ 0 & e^{-i\psi/2} \end{pmatrix}, \quad (17)$$

where ψ is the Wigner angle and z is an arbitrary complex number.

Following [9], we now compute S by the right hand expression in Eq.(16), using the expression for A in Eq.(15) for an arbitrary LT. The $SL(2,C)$ element A_k is obtained by solving $K = A_k \tilde{K} A^\dagger_k$ and is given by

$$A_k = \frac{1}{\sqrt{2(1+n^{\hat{3}})}} \begin{pmatrix} 1+n^{\hat{3}} & -n_- \\ n_+ & 1+n^{\hat{3}} \end{pmatrix} \begin{pmatrix} \sqrt{k^{\hat{0}}} & 0 \\ 0 & 1/\sqrt{k^{\hat{0}}} \end{pmatrix}, \quad (18)$$

corresponding to the product of a boost in the z-direction taking $\tilde{k}^{\hat{0}} = 1 \to k^{\hat{0}}$, and a rotation taking $\hat{\boldsymbol{z}} \to \hat{\boldsymbol{k}}$.

Lastly, we can compute the corresponding element of $A_{k'} \in SL(2,C)$ which transforms $K \to K'$ by an arbitrary LT of the form Eq.(15). Here K' has the same form as Eq.(14) but with all quantities primed. The result is [9]

$$\begin{aligned} K' &= k'^{\hat{0}} \begin{pmatrix} 1+n'^{\hat{3}} & n'_- \\ n'_+ & 1-n'^{\hat{3}} \end{pmatrix} \\ &= A_{k'}\, K\, A^\dagger_{k'} \equiv k^{\hat{0}} \begin{pmatrix} b & c^* \\ c & a-b \end{pmatrix}, \end{aligned} \quad (19)$$

where we have defined

$$\begin{aligned} a &= (|\alpha|^2+|\gamma|^2)(1+n^{\hat{3}}) + (|\beta|^2+|\delta|^2)(1-n^{\hat{3}}) \\ &+ (\alpha\beta^*+\gamma\delta^*)n_- + (\alpha^*\beta+\gamma^*\delta)n_+ = a^* \\ b &= |\alpha|^2(1+n^{\hat{3}}) + |\beta|^2(1-n^{\hat{3}}) + \alpha\beta^* n_- + \alpha^*\beta n_+ = b^* \\ c &= \alpha^*\gamma(1+n^{\hat{3}}) + \beta^*\delta(1-n^{\hat{3}}) + \beta^*\gamma n_- + \alpha^*\delta n_+. \end{aligned} \quad (20)$$

From Eq.(19) and Eq.(20) we can deduce the transformation of the photon 4-momentum as

$$k'^{\hat{0}} = \frac{a}{2}k^{\hat{0}}, \quad n'^{\hat{3}} = \frac{2b}{a} - 1, \quad n'_+ = \frac{2c}{a} \quad (21)$$

where the quantities a,b,c in Eq.(20) depend only on the LT parameters $\alpha,\beta,\gamma,\delta$ of A_k and the *direction* of the photon $n_\pm, n^{\hat{3}}$, but not on the photon frequency $k^{\hat{0}}$.

Forming the inverse of Eq.(19) and substituting it and Eq.(18) into the right hand side of Eq.(16), and subsequently equating the result to Eq.(17), yields

$$\begin{aligned} e^{i\psi(\Lambda,k)/2} &= \frac{1}{a\sqrt{b(1+n^{\hat{3}})}} \Big\{ [\alpha(1+n^{\hat{3}}) + \beta n_+]b \\ &+ [\gamma(1+n^{\hat{3}}) + \delta n_+]c^* \Big\}. \end{aligned} \quad (22)$$

Note that the expression for the arbitrary complex number z is not needed since it does not enter into the expression for the transformation of the photon helicity state Eq.(5).

Finally, to relate the entries of Eq.(15) to the infinitesimal LLT $\lambda^{\hat{a}}{}_{\hat{b}}$, we expand $K' = AKA^\dagger$ in terms of $k^{\hat{a}}$ and $k'^{\hat{a}} \equiv \Lambda^{\hat{a}}{}_{\hat{b}}\, k^{\hat{b}}$ and expand $\Lambda^{\hat{a}}{}_{\hat{b}}$ as in Eq.(12). Multiplying by a general Pauli matrix and using the relation $\frac{1}{2}tr(\sigma_{\hat{a}}\sigma_{\hat{b}}) = \delta_{\hat{a}\hat{b}}$ yields the expression

$$\lambda^{\hat{a}}{}_{\hat{b}} = \frac{1}{2}\eta^{\hat{a}\hat{c}}\, tr\left(\sigma_{\hat{b}}\sigma_{\hat{c}}A + \sigma_{\hat{c}}\sigma_{\hat{b}}A^\dagger\right), \quad (23)$$

where tr is the matrix trace. Since we are interested in an infinitesimal LLT we further expand A as

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv I + \tilde{A}\, d\xi, \quad \tilde{A} = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & -\tilde{\alpha} \end{pmatrix}, \quad (24)$$

which satisfies $\det(A) = 1$ to first order in $d\xi$. A straightforward calculation leads to

$$\begin{aligned} \tilde{\alpha} &= \frac{1}{2}\left(\lambda^{\hat{0}}{}_{\hat{3}} - i\lambda^{\hat{1}}{}_{\hat{2}}\right), \\ \tilde{\beta} &= \frac{1}{2}\left[\left(\lambda^{\hat{0}}{}_{\hat{1}} + \lambda^{\hat{3}}{}_{\hat{1}}\right) - i\left(\lambda^{\hat{0}}{}_{\hat{2}} + \lambda^{\hat{2}}{}_{\hat{3}}\right)\right] \\ \tilde{\gamma} &= \frac{1}{2}\left[\left(\lambda^{\hat{0}}{}_{\hat{1}} - \lambda^{\hat{3}}{}_{\hat{1}}\right) + i\left(\lambda^{\hat{0}}{}_{\hat{2}} - \lambda^{\hat{2}}{}_{\hat{3}}\right)\right], \end{aligned} \quad (25)$$

with $\lambda^{\hat{a}}{}_{\hat{b}}(x) = \chi^{\hat{a}}{}_{\hat{b}}(x)$ given in terms of the observer's tetrad by Eq.(11). Using the above expressions for $\tilde{A}$ we can form the entries of A, and using the expression for a,b,c from Eq.(20) determine the infinitesimal Wigner

rotation angle $\tilde{\psi}(\Lambda, \vec{n})$ when we expand Eq.(22) to $\mathcal{O}(d\xi)$ as

$$e^{i\psi(\Lambda,k)/2} \sim 1 + i\tilde{\psi}(\Lambda, k)\, d\xi/2. \tag{26}$$

Finite Wigner rotations can be built up as a time ordered integration of infinitesimal Wigner rotations over the geodesic trajectory $x(\xi)$ of the photon (obtained by solving Eq.(9)) via

$$\exp\Big(i\psi(\Lambda,\vec{n})/2\Big) = T\exp\left[i\int \tilde{\psi}\Big(\Lambda,\vec{n}(\xi)\Big)\, d\xi/2\right] \tag{27}$$

where $\vec{n}(\xi) = \vec{n}(x(\xi))$, $\Lambda^{\mu}{}_{\nu}(\xi) = \Lambda^{\mu}{}_{\nu}(x(\xi))$ and T is the time order operator.

B. Examples of the Wigner rotation angle in the Schwarzschild metric

We now consider some specific examples of the Wigner rotation angle $\psi(\Lambda, k)$ as computed from Eq.(20) and Eq.(22) in the static, spherically symmetric Schwarzschild spacetime, and compare and contrast the results with SR, which holds locally at each spacetime point. The Schwarzschild metric is given by

$$ds^2 = (1-r_s/r)\, c^2 dt^2 - \frac{dr^2}{(1-r_s/r)} + r^2\,(d\theta^2 + r^2\,\sin^2\theta\, d\phi^2) \tag{28}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius of the central gravitating object of mass M (e.g. for the Earth $r_{s\oplus} = 0.89$ cm, and for the Sun $r_{s\odot} = 2.96$ km). Henceforth, we use units where $G = c = 1$. Since the metric is independent of ϕ, orbital angular momentum is conserved [12], so without loss of generality we consider photon orbits in the equatorial plane $\theta = \pi/2$.

1. *Radially infalling photons*

The 4-momentum for a radially in-falling photon (satisfying $\nabla_{\mathbf{k}}\,\mathbf{k} = 0$) with zero orbital angular momentum is given by

$$\begin{aligned} k^{\mu}(x) &\equiv \big(k^t(x),\, k^r(x),\, k^{\theta}(x),\, k^{\phi}(x)\big) \\ &= \omega\,\big(1/(1-r_s/r),\, -1, 0,\, 0\big), \end{aligned} \tag{29}$$

satisfying $\mathbf{k}\cdot\mathbf{k} \equiv g_{\mu\nu}\, k^{\mu}\, k^{\nu} = 0$. The constant (of the motion) ω is the frequency of the photon as measured by a stationary observer (discussed below) at spatial infinity ($r \to \infty$).

We now consider several different types of (massive) observers defined by stating their associated tetrads. An obvious first choice is to consider a *stationary* observer who sits at fixed spatial coordinates $(r,\, \theta,\, \phi\,)$ whose tetrad is given by

$$\begin{aligned} (e^{stat}_{\hat{0}})^{\mu}(x) &= \Big(1/(1-r_s/r)^{1/2}, 0, 0, 0\Big) = \mathbf{e}^{stat}_{\hat{t}}, \\ (e^{stat}_{\hat{3}})^{\mu}(x) &= \Big(0, (1-r_s/r)^{1/2}, 0, 0\Big) = \mathbf{e}^{stat}_{\hat{r}}, \\ (e^{stat}_{\hat{1}})^{\mu}(x) &= (0, 0, 1/r, 0) = \mathbf{e}^{stat}_{\hat{\theta}}, \\ (e^{stat}_{\hat{2}})^{\mu}(x) &= (0, 0, 0, 1/r) = \mathbf{e}^{stat}_{\hat{\phi}}, \end{aligned} \tag{30}$$

Since we are considering motion in the equatorial plane, we have oriented our coordinate system so that the local $\hat{3}$ axis points along the (increasing) radial direction, the $\hat{2}$ axis points along the (increasing) ϕ direction, then the $\hat{1}$ axis direction (normal to the equatorial plane) points in the (increasing) θ direction. Note that the world components of the tetrad vectors $e_{\hat{a}}{}^{\alpha}(x)$ are given in the order $\alpha = (0, 1, 2, 3) \leftrightarrow (t, r, \theta, \phi)$, while the ordering of our observer's local axes are given in the order $\hat{a} = (\hat{0}, \hat{1}, \hat{2}, \hat{3}) \leftrightarrow (\hat{t}, \hat{\theta}, \hat{\phi}, \hat{r})$ corresponding to the observer's local time and $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ axes.

A stationary observer must exert an acceleration $\mathbf{a}$ defined by $\mathbf{a} = \nabla_{\mathbf{u}}\,\mathbf{u}$ in order to oppose the gravitational attraction of the central mass M and remain at a fixed spatial location. (Note that in SR a stationary observer undergoes zero acceleration since there is no gravitational field, i.e. $M = 0$). In general, the magnitude of the local acceleration experienced by the observer in his local frame is $a \equiv ||\mathbf{a}|| \equiv (-\mathbf{a}\cdot\mathbf{a})^{1/2}$ (where the minus sign results from the fact that $\mathbf{a}$ is a spacelike vector). For the stationary observer, the local acceleration is given by $a^{stat} = (M/r^2)\,(1-r_s/r)^{-1/2}$, which approaches the usual Newtonian form of M/r^2 as $r \to \infty$. The fact that $\lim_{r\to r_s} a^{stat} = \infty$ indicates that for $r < r_s$, i.e. inside the event horizon of the black hole, the observer can no longer remain stationary and is inexorably drawn into the singularity of the black hole.

From the above tetrad we can compute the local components $k^{\hat{a}}(x)$ of the photon's 4-momentum $k^{\mu}(x)$ as measured in the observer's local frame with $k^{\hat{a}}(x) = e^{\hat{a}}{}_{\mu}(x)\, k^{\mu}(x)$. Here, $e^{\hat{a}}{}_{\mu}(x)$ is the inverse transpose of the matrix of tetrad vectors $e_{\hat{a}}{}^{\mu}(x)$. The unit vector $n^{\hat{i}}$, used in Eq.(20) and Eq.(22), which describes the direction of the photon as measured in the observer's local frame is given by $n^{\hat{i}} = k^{\hat{i}}/(k^{\hat{i}}\,k_{\hat{i}})^{1/2}$ where $k^{\hat{i}}\,k_{\hat{i}} = \sum_{\hat{i}=\hat{1}}^{\hat{3}} (k^{\hat{i}})^2$ is the ordinary Euclidean flat spacetime dot product of the spatial 3-vector portion $k^{\hat{i}}$ of the local 4-vector $k^{\hat{a}}$. For the radially infalling photon that we consider, we have $n^{\hat{i}} = (0,\, 0,\, -1\,)$.

The general formula for the Wigner rotation angle $\psi(\Lambda, \vec{n})$ is given in Eq.(22). Since all relevant quantities in this formula are a function of the spacetime point x, we are interested in computing the infinitesimal Wigner rotation angle $\tilde{\psi}(\Lambda, k)$ as defined in Eq.(26). This requires the infinitesimal version of the entries $\{\alpha, \beta, \gamma, \delta\}$ of the $SL(2, C)$ matrix A in Eq.(24) which describes the LLT. We defined these infinitesimal entries as $\{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}\}$ through the $SL(2, C)$ matrix $\tilde{A}$ in the same equation,

which described the infinitesimal LLT. From Eq.(25), $\{\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta}\}$ are related to the 4×4 infinitesimal LLT $\lambda^{\hat{a}}{}_{\hat{b}}$, which from Eq.(13) is equal to the matrix $\chi^{\hat{a}}{}_{\hat{b}}$ describing the rotation of the tetrads. Finally, $\chi^{\hat{a}}{}_{\hat{b}}$ can be computed from Eq.(11) given the observer's tetrad and the photon 4-momentum. Thus, combining all these infinitesimal contributions into the right hand side of Eq.(22) and equating this to Eq.(26) allows us to extract the Wigner rotation angle $\psi(\Lambda,\vec{n})$ to first order in $d\xi$.

Applying the above prescription to the radially infalling photon and the stationary observer leads to the result $\psi(\Lambda,\vec{n})=0$. This is an expected result since as the photon moves along its radial geodesic it is continually boosted in the same direction. As in the flat spacetime case, an stationary observer in the path of the photon detects no Wigner rotation of the photon's polarization [9].

As discussed above, the stationary observer in a static metric undergoes a non-zero acceleration to remain at a fixed spatial location. The general relativistic observer that is locally analogous to the inertial (constant velocity) observer of SR is the *freely falling frame* (FFF). The FFF is defined by the tetrad $\mathbf{e}_{\hat{a}}(x)$ satisfying the condition

$$\nabla_{\mathbf{u}}\,\mathbf{e}_{\hat{a}}=0. \tag{31}$$

Since $\mathbf{e}_{\hat{0}}(x)=\mathbf{u}(x)$ is the 4-velocity of the observer's geodesic, the $\hat{a}=\hat{0}$ equation in Eq.(31) is just the geodesic equation, stating that the FFF observer experience zero local acceleration $||\mathbf{a}^{FFF}||=0$. The remaining $\hat{a}=\hat{i}$ equations state that the spatial tetrad axes $\mathbf{e}_{\hat{i}}(x)$ are parallel transported along the observer's geodesic.

In the Schwarzschild metric, the tetrad for a radially FFF observer is given by

$$\begin{aligned}
(e^{FFF}_{\hat{0}})^{\mu}(x) &= \left((1-r_s/r)^{-1},-(r_s/r)^{1/2},0,0\right)=\mathbf{e}^{FFF}_{\hat{t}},\\
(e^{FFF}_{\hat{3}})^{\mu}(x) &= \left(-(r_s/r)^{1/2}\,(1-r_s/r)^{-1},1,0,0\right)=\mathbf{e}^{FFF}_{\hat{r}},\\
(e^{FFF}_{\hat{1}})^{\mu}(x) &= (0,0,1/r,0)=\mathbf{e}^{FFF}_{\hat{\theta}},\\
(e^{FFF}_{\hat{2}})^{\mu}(x) &= (0,0,0,1/r)=\mathbf{e}^{FFF}_{\hat{\phi}},
\end{aligned} \tag{32}$$

Following the above prescription with the FFF tetrad, we again find that $\psi(\Lambda,\vec{n})=0$. Since SR holds at each spacetime point, we could have invoked Einstien's equivalence principle to deduce this last result.

2. *Photon with nonzero angular momentum*

A general photon orbit in the Shcwarzschild metric obeys the radial "energy" equation [13]

$$\frac{1}{b^2_{ph}}=\left(\frac{dr}{d\xi}\right)^2+W_{eff}(r),\quad W_{eff}(r)=\frac{1}{r^2}\left(1-\frac{r_s}{r}\right), \tag{33}$$

where ξ is the affine parameter along the photon geodesic, and $k^{\mu}=dx^{\mu}/d\xi$. In Eq.(33), the quantity $1/b^2_{ph}$ acts as an effective energy. Here $b_{ph}=|l_{ph}/e_{ph}|$ is the ratio of the orbital angular momentum l_{ph} and the energy e_{ph} of the photon, both of which are constant since the Schwarzschild metric is independent of t and ϕ, respectively. For $r\gg r_s$ the quantity b has the interpretation of the impact parameter of the photon with respect to M situated at $r=0$. The most general infalling photon geodesic (starting at spatial infinity) in the equatorial plane ($k^{\theta}=d\theta/d\xi=0$) is given by

$$k^{\mu}(x)=\left(\frac{1}{(1-r_s/r)},-\left(1-\frac{b^2_{ph}(1-r_s/r)}{r^2}\right)^{1/2},0,\frac{b_{ph}}{r^2}\right), \tag{34}$$

which reduces to the radial infalling photon in the equatorial plane Eq.(29), for $b_{ph}\to 0$. Unlike the flat spacetime case of SR, the orbit of photon in Eq.(33) is curved, and there even exits an unstable circular orbit for $b^2_{ph}=27M^2$ for which W_{eff} has a maximum value. For stationary metrics, one can interpret the the bending of light in an optical-mechanical analogy in which the metric can be viewed as a spatially varying index of refraction, that takes on a value of unity at spatial infinity and is infinite at the event horizon [14]. For $1/b^2_{ph}<1/(27M^2)$ there is a turning point in the photon orbit, and the photon will again escape to infinity. For $1/b^2_{ph}>1/(27M^2)$ the photon spirals into the event horizon and is captured by the black hole.

An observer (massive) traveling on a geodesic in the equatorial plane satisfies the corresponding radial "energy" equation [13]

$$\begin{aligned}
\mathcal{E}\equiv\frac{e^2_{obs}-1}{2} &= \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+V_{eff}(r),\\
V_{eff}(r) &= -\frac{M}{r}+-\frac{l_{obs}}{2r^2}-\frac{Ml^2_{obs}}{r^3},
\end{aligned} \tag{35}$$

where the constants of the motion $e_{obs}=(1-r_s/r)\,dt/d\tau$ and $l_{obs}=r^2 sin^2\theta\,d\phi/d\tau$ (here $\theta=\pi/2$) can be interpreted, at large r, as the observer's energy per unit mass and orbital angular momentum per unit mass. Here τ is the observer's proper time defined from the metric Eq.(28) as $c\,d\tau=ds$. In general, the FFF satisfying $\nabla_{\mathbf{u}}\,\mathbf{e}_{\hat{a}}=0$ depends separately on both e_{obs} and l_{obs}. To keep the algebra manageable, in the following we will consider the observer's geodesic to lie in the equatorial plane ($u^{\theta}=d\theta/d\tau=0$ with $e_{obs}=1$ but arbitrary orbital angular momentum l_{obs}. This FFF tetrad, which we denote by $\mathbf{e}^{FFF(l)}_{\hat{a}}(x)$ is given by

$$
\begin{aligned}
(e_{\hat{0}}^{FFF(l)})^\mu(x) &= \left(\frac{1}{(1-r_s/r)}, u^r, 0, \frac{l_{obs}}{r^2}\right) = \mathbf{e}_{\hat{t}}^{FFF(l)}, \\
(e_{\hat{3}}^{FFF(l)})^\mu(x) &= \left(-\sqrt{\frac{r_s}{r}}\,\frac{\cos\Phi(r)}{(1-r_s/r)}, -\sqrt{\frac{r_s}{r}}\,u^r\,\cos\Phi(r) - \frac{l_{obs}\,(1-r_s/r)\,\sin\Phi(r)}{\sqrt{r_s r}}, 0, \frac{l_{obs}\,\cos\Phi(r)}{\sqrt{r_s r^3}} - \frac{u^r\,\sin\Phi(r)}{\sqrt{r_s r}}\right) = \mathbf{e}_{\hat{r}}^{FFF(l)}, \\
(e_{\hat{1}}^{FFF(l)})^\mu(x) &= (0,0,1/r,0) = \mathbf{e}_{\hat{\theta}}^{FFF(l)}, \\
(e_{\hat{2}}^{FFF(l)})^\mu(x) &= \left(-\sqrt{\frac{r_s}{r}}\,\frac{\sin\Phi(r)}{(1-r_s/r)}, \frac{l_{obs}\,(1-r_s/r)\,\cos\Phi(r)}{\sqrt{r_s r}} + \sqrt{\frac{r}{r_s}}\,u^r\,\sin\Phi(r), 0, -\frac{u^r\,\cos\Phi(r)}{\sqrt{r_s r}} + \frac{l_{obs}\,\sin\Phi(r)}{\sqrt{r_s r^3}}\right) = \mathbf{e}_{\hat{\phi}}^{FFF(l)},
\end{aligned}
\tag{36}
$$

where the radial component of the observer's 4-velocity $u^r = dr/d\tau$ and the angle of rotation $\Phi(r)$ of the spatial tetrads in the equatorial plane are given by

$$
\begin{aligned}
u^r(r) &= -\left(\frac{r_s}{r} - \frac{l_{obs}^2}{r^2}\,(1-r_s/r)\right)^{1/2}, \\
\frac{d\Phi(r)}{dr} &= -\frac{l_{obs}}{2\,r^2\,u^r(r)}.
\end{aligned}
\tag{37}
$$

The photon and observer geodesics given by Eq.(34) and Eq.(36), respectively, both lie in the equatorial plane ($\theta = \pi/2$). From the discussion in the appendix of the Wigner rotation in flat spacetime, we can invoke the EP to associate the observer's local (spatial tetrad) axes $(\hat{1}, \hat{2}, \hat{3})$ with the inertial axes $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ used in the appendix to discuss the Wigner rotation in the flat spacetime of SR. For both the photon and observer geodesics in the equatorial plane $\hat{2}$-$\hat{3}$ ($\mathbf{e}_{\hat{\phi}}$ - $\mathbf{e}_{\hat{r}}$) corresponding to the SR $\hat{\mathbf{y}}$-$\hat{\mathbf{z}}$ plane used in the appendix, the Wigner rotation is identically zero, $\psi(\Lambda, \vec{n}) = 0$ (see the discussion in the appendix following Fig.(2)). This is borne out by a GR calculation utilizing Eq.(34) and Eq.(36).

Following the appendix, to obtain a non-zero Wigner angle in flat spacetime, we considered the situation with the photon moving in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ with the observer moving in the $\hat{\mathbf{y}}$-$\hat{\mathbf{z}}$ plane (see the discussion in the appendix of Fig.(4)). Invoking the EP this corresponds in the GR case to the photon's geodesic being in the $\hat{1}$-$\hat{2}$ or $\mathbf{e}_{\hat{\theta}}$-$\mathbf{e}_{\hat{\phi}}$ plane, while the observer's geodesic remains in the equatorial plane, $\hat{2}$-$\hat{3}$, or $\mathbf{e}_{\hat{\phi}}$ - $\mathbf{e}_{\hat{r}}$ plane. We can modify the photon geodesic lying in the equatorial plane $\theta = \pi/2$ Eq.(34), to a geodesic lying in the plane $\phi = \pi/2$ with the expression

$$
\begin{aligned}
k^\mu(x) &= \left(\frac{1}{(1-r_s/r)}, -\left(1 - \frac{b_{ph}^2(1-r_s/r)}{r^2}\right)^{1/2}, \frac{b_{ph}}{r^2}, 0\right), \\
&\approx \left(\frac{1}{(1-r_s/r)}, -1, \frac{b_{ph}}{r^2}, 0\right) + O(b_{ph}^2).
\end{aligned}
\tag{38}
$$

In the last line of Eq.(38) we have expanded the photon 4-vector to first order in b_{ph}, to keep the algebra manageable in the following calculation. Similarly, expanding Eq.(36) to first order in l_{obs} we obtain

$$
\begin{aligned}
(e_{\hat{0}}^{FFF(l)})^\mu(x) &= \left(\frac{1}{(1-r_s/r)}, -\sqrt{\frac{r_s}{r}}, 0, \frac{l_{obs}}{r^2}\right) = \mathbf{e}_{\hat{t}}^{FFF}, \\
(e_{\hat{3}}^{FFF(l)})^\mu(x) &= \left(\sqrt{\frac{r_s}{r}}\,\frac{1}{(1-r_s/r)}, 1, 0, -\frac{2l_{obs}}{\sqrt{r_s r^3}}\right) = \mathbf{e}_{\hat{r}}^{FFF}, \\
(e_{\hat{1}}^{FFF(l)})^\mu(x) &= (0,0,1/r,0) = \mathbf{e}_{\hat{\theta}}^{FFF}, \\
(e_{\hat{2}}^{FFF(l)})^\mu(x) &= \left(\frac{-l_{obs}}{r}\,\frac{1}{(1-r_s/r)}, \frac{l_{obs}\,(2-r_s/r)}{\sqrt{r_s r}}, 0, \frac{1}{r}\right) \\
&= \mathbf{e}_{\hat{\phi}}^{FFF},
\end{aligned}
\tag{39}
$$

which satisfies the FFF tetrad conditions Eq.(31) and $\mathbf{e}\cdot\mathbf{g}\cdot\mathbf{e}^T = \boldsymbol{\eta}$ to correction terms of $O(l_{obs}^2)$. Computing $\tilde{\psi}(\Lambda, k)$ in Eq.(26) we find a non-zero (infinitesimal) Wigner rotation angle.

$$
\tilde{\psi}(\Lambda, k) = \frac{b_{ph}\,l_{obs}}{r^3}\,\left(1 + \frac{3}{2}\sqrt{\frac{r}{r_s}} - \frac{5}{4}\sqrt{\frac{r_s}{r}}.\right) \tag{40}
$$

Note that the above result is proportional to both b_{ph} and l_{obs}. If $b_{ph} = 0$, the photon would be radial in the plane $\phi = \pi/2$ and would therefore only intersect the observer's curved geodesic in the equatorial plane $\theta = \pi/2$ at $r = 0$. Similarly, if $l_{obs} = 0$, the observer's geodesic would be radial in the equatorial plane, and intersect the photons curved geodesic again at $r = 0$. For both b_{ph} and l_{obs} non-zero we have the trajectories as illustrated in Fig.(1). Note that in general the photon and and observer's geodesic intersect at one spacetime point (the two-toned circle in the figure). The observer at this instant in time and this location, observing the photon traversing his local laboratory, will measure the nonzero infinitesimal Wigner angle in Eq.(40).

V. GENERAL FRAMES IN WHICH THE WIGNER ROTATION ANGLE IS ZERO

A. Conditions

We now explore the condition under which the observer will observe zero Wigner rotation of the photon's

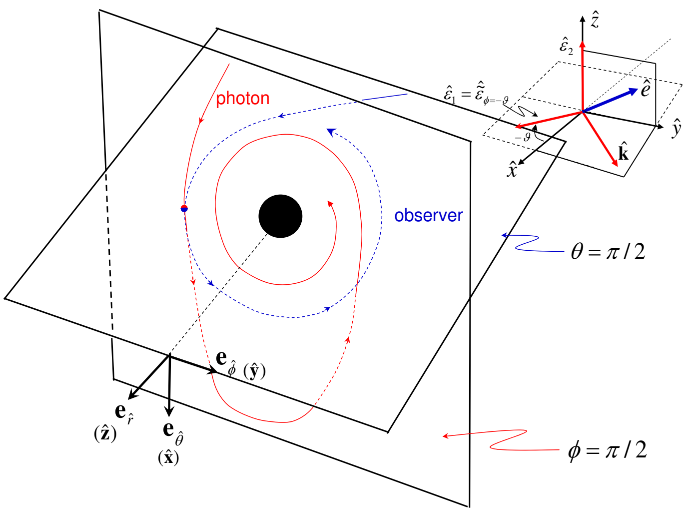

FIG. 1: (Color online) Example of a nonzero Wigner angle measured by an observer in Schwarzschild spacetime. In Schwarzschild coordinates $x^\alpha = (t, r, \theta, \phi)$, the photon geodesic lies in the plane $\phi = \pi/2$, while the observer's geodesic lies in the $\theta = \pi/2$ equatorial plane. The photon geodesic has a nonzero impact parameter b_{ph}, while the observer's geodesic has nonzero angular momentum l_{obs}. The two-toned colored circle shows the intersection of photon and observer geodesics. Compare with the SR flat spacetime case illustrated in Fig.(4), which by the EP, holds at the intersection point.

polarization. From Eq.(13) the change $\delta k_{\hat{a}}(x)$ in the local components of the photon wavevector is zero if $\lambda_{\hat{a}}{}^{\hat{b}}(x) = \chi_{\hat{a}}{}^{\hat{b}}(x) = 0$. From Eq.(11) a sufficient condition for this to occur is

$$\nabla_{\mathbf{k}}\, e_{\hat{a}}{}^{\nu}(x) = 0. \tag{41}$$

In the case of massive particles considered by Alsing *et al* [3], the photon momentum $\mathbf{k}$ would be replaced by the massive particle's 4-velocity $\mathbf{u} = \mathbf{e}_{\hat{0}}$. The corresponding condition $\nabla_{\mathbf{u}}\, e_{\hat{a}}{}^{\nu}(x) = 0$ defines the instantaneous non-rotating rest frame of the particle traveling on a geodesic (zero acceleration - if one ignores the particle's spin), i.e. the observer's local laboratory rides along with the passing particle. Since there is no rest frame for a photon, Eq.(41) describes something different, and it is not immediately obviously that a solution for the observer's local laboratory, described by the tetrad $e_{\hat{a}}{}^{\nu}(x)$, exists.

We have found solutions to Eq.(41), which describe a class of observers situated at each spacetime point x, which are in fact Fermi-Walker frames (FWF). The FWF is the instantaneous non-rotating rest frame of a particle experiencing arbitrary non-zero acceleration $\mathbf{a} = \nabla_{\mathbf{u}}\, \mathbf{u}$, and is defined by the following equation

$$\nabla_{\mathbf{u}}\, \mathbf{s} = (\mathbf{u}\cdot\mathbf{s})\,\mathbf{a} - (\mathbf{a}\cdot\mathbf{s})\,\mathbf{u}. \tag{42}$$

A vector $\mathbf{s}$ satisfying Eq.(42) is said to be *Fermi Walker transported*. If $\mathbf{s} = \mathbf{u}$, Eq.(42) reproduces the definition of the acceleration $\mathbf{a}$. In general, $\mathbf{s}$ is any one of the three orthonormal spatial axes $e_{\hat{a}}{}^{\hat{i}}(x)$ of the observer's tetrad that is orthogonal to $\mathbf{u}$, and Eq.(42) is a generalization of parallel transport when $\mathbf{a} \neq 0$. Equation (42) defines a locally *non-rotating* frame in the sense that if $\mathbf{s}$ is orthogonal to the instantaneous osculating plane defined by $\mathbf{u}$ and $\mathbf{a}$ then $\nabla_{\mathbf{u}}\, \mathbf{s} = 0$, i.e. $\mathbf{s}$ is parallel transported along the world line with tangent $\mathbf{u}$.

B. Sufficient condition for existence of zero Wigner rotation angle frames

That solutions of Eq.(41) can be found that also satisfy Eq.(42) is not readily obvious since the later FW transport equation is a statement solely about the observer and therefore is independent of the photon 4-momentum $\mathbf{k}$. To show that solutions of Eq.(41) are compatible with Eq.(42) let $\mathbf{s}$ be any of the three orthonormal spatial vectors of the tetrad, and $\mathbf{u} = \mathbf{e}_{\hat{t}}$ be the observer's 4-velocity. Hence, $\mathbf{u}\cdot\mathbf{u} = 1$, $\mathbf{s}\cdot\mathbf{s} = -1$, and $\mathbf{u}\cdot\mathbf{s} = 0$. Equation (27) is then restated as

$$\nabla_{\mathbf{k}} e_{\hat{a}}{}^{\nu}(x) = 0, \quad \Rightarrow \quad \nabla_{\mathbf{k}}\mathbf{u} = 0,\ \nabla_{\mathbf{k}}\mathbf{s} = 0. \tag{43}$$

As a integrability condition, we take $\nabla_{\mathbf{k}}$ of the FW transport equation Eq.(42) and use Eq.(43) repeatedly along with the orthogonality of $\mathbf{u}$ and $\mathbf{s}$, noting that $\mathbf{a} = \nabla_{\mathbf{u}}\mathbf{u}$. This yields the equation

$$\nabla_{\mathbf{k}}\, \nabla_{\mathbf{u}}\, \mathbf{s} + [(\nabla_{\mathbf{k}} \nabla_{\mathbf{u}} \mathbf{u})\cdot\mathbf{s}]\,\mathbf{u} = 0. \tag{44}$$

Projecting Eq.(44) onto the the particle's 4-velocity by taking its dot product with $\mathbf{u}$ yields

$$(\nabla_{\mathbf{k}}\, \nabla_{\mathbf{u}}\, \mathbf{s})\cdot\mathbf{u} + (\nabla_{\mathbf{k}}\, \nabla_{\mathbf{u}}\, \mathbf{u})\cdot\mathbf{s} = 0. \tag{45}$$

Equation (45) can be obtained independently by differentiating the orthogonality condition $\mathbf{s}\cdot\mathbf{u} = 0$, first with respect to $\nabla_{\mathbf{u}}$ and then with respect to $\nabla_{\mathbf{k}}$ and making repeated use of Eq.(43).

Similarly, by projecting Eq.(44) onto an arbitrary spatial vector $\mathbf{s}'$ of the tetrad we obtain

$$\left(\nabla_{\mathbf{k}}\, \nabla_{\mathbf{u}}\, \mathbf{s}\right)\cdot\mathbf{s}' = 0. \tag{46}$$

Since we could have equivalently written down Eq.(44) using $\mathbf{s}'$ and then subsequently projected onto $\mathbf{s}$, Eq.(46) must also hold with these two vectors reversed, i.e.

$$\left(\nabla_{\mathbf{k}}\, \nabla_{\mathbf{u}}\, \mathbf{s}'\right)\cdot\mathbf{s} = 0. \tag{47}$$

Since $\mathbf{s}'\cdot\mathbf{s} = -1$ if $\mathbf{s}' = \mathbf{s}$, and zero otherwise, applying first $\nabla_{\mathbf{u}}$ and then $\nabla_{\mathbf{k}}$ to the relation $\mathbf{s}'\cdot\mathbf{s} =$ constant, and again repeatedly using Eq.(43), yields

$$\left(\nabla_{\mathbf{k}}\, \nabla_{\mathbf{u}}\, \mathbf{s}\right)\cdot\mathbf{s}' + \left(\nabla_{\mathbf{k}}\, \nabla_{\mathbf{u}}\, \mathbf{s}'\right)\cdot\mathbf{s} = 0. \tag{48}$$

One possible solution of Eq.(48) is that each separate term is identically zero as in Eq.(46) and Eq.(47). This

shows that a solution of Eq.(41) has the FW transport equation Eq.(42) as a sufficient condition, though not a necessary condition. For example, in Schwarzschild spacetime we can explicitly solve for the tetrad for a radially accelerating FWF observer for case of a radially infalling photon, both in the equatorial plane such that $\psi = 0$ (though, as discussed in the appendix, the Wigner angle is identically zero in this case when both the photon and observer geodesics are in the equatorial plane). Finally, note that nowhere in the above argument have we made use of the fact that $\mathbf{k}$ is a null vector.

VI. ENTANGLEMENT CONSIDERATIONS

A. Photon Helicity States

As discussed in Section III A, in CST the transformation of a photon state $|\mathbf{k},\lambda\rangle$ of pure helicity λ has the same form as the SR flat spacetime form

$$U(\Lambda)\,|\mathbf{k},\lambda\rangle = e^{i\,\lambda\,\psi(\Lambda,\vec{n})}|\mathbf{k}',\lambda\rangle, \tag{49}$$

if we interpret $\mathbf{k} = (k^0, |\vec{k}|\vec{n})$, $(\vec{n}\cdot\vec{n} = 1)$ as the the photon wavevector as measured by the observer described by the tetrad $\mathbf{e}_{\hat{a}}(x)$, i.e. with components $\mathbf{k} \leftrightarrow k^{\hat{a}}(x) = e^{\hat{a}}{}_{\alpha}(x)\,k^{\alpha}(x)$, and Λ a LLT as given in Eq.(12)

$$\begin{aligned} k^{\hat{a}}(x) &\rightarrow k'^{\hat{a}}(x) \equiv k^{\hat{a}}(x) + \delta k^{\hat{a}}(x) \\ &= \Lambda^{\hat{a}}{}_{\hat{b}}(x)\,k^{\hat{b}}(x) = \left(\delta^{\hat{a}}{}_{\hat{b}} + \lambda^{\hat{a}}{}_{\hat{b}}(x)\,d\xi\right)\,k^{\hat{b}}(x), \end{aligned}$$

for the infinitesimal motion of the photon along its geodesic from $x^\mu \rightarrow x^{'\mu} = x^\mu + k^\mu(x)\,\delta\xi$. As discussed earlier, the Wigner rotation angle is a function of the LLT $\lambda^{\hat{a}}{}_{\hat{b}}(x)$, the direction of propagation $\vec{n}$ of the photon, but not its frequency $\omega = |\vec{k}|$, i.e. $\psi = \psi(\Lambda,\vec{n})$.

Eq.(49) holds for an infinitesimal LLT with $\psi(\Lambda,\vec{n})$ computed from Eq.(22) and Eq.(26). For finite motion of the photon along its geodesic from $x^\mu \rightarrow x^{'\mu}$ the time ordered expression for the the Wigner angle in Eq.(27) must be employed, and Eq.(49) generalized to

$$\begin{aligned} U(\Lambda)\,|k^{\hat{a}}(x),\,\lambda\rangle &= T\,[e^{i\lambda\int\tilde{\psi}(\Lambda(\xi),\,\vec{n}(\xi))d\xi}] \\ &\times\ |\,T\,[e^{\int\tilde{\lambda}^{\hat{a}}{}_{\hat{b}}(\Lambda(x(\xi)),\,k(\xi))d\xi}]\,k^{\hat{b}}(x),\,\lambda\rangle. \end{aligned} \tag{50}$$

The term outside the ket is the time ordered product of the infinitesimal Winger rotation $\tilde{\psi}(\Lambda(\xi),\,\vec{n}(\xi))$ along the photon's geodesic parameterized by ξ, while the term inside the ket is the time ordered product of the LLT $\tilde{\lambda}^{\hat{a}}{}_{\hat{b}}\big(\Lambda(x(\xi)),\,k(\xi)\big)$ along the observer's geodesic through which the photon passes. The time ordered integration is over the extent of ξ for which the photon passes through the observer's local laboratory.

A more relevant way to interpret Eq.(50) is as follows. Rather than considering a single observer with tetrad $\mathbf{e}_{\hat{a}}(x)$ and following his motion through spacetime , we consider $\mathbf{e}_{\hat{a}}(x)$ as describing an infinite set of observers distributed throughout spacetime at points x (e.g. the class of stationary observers with tetrad Eq.(30), each at a fixed spatial location). Then, as the photon passes through the local laboratories of this set of observers with tetrad $\mathbf{e}_{\hat{a}}(x)$, $\tilde{\lambda}^{\hat{a}}{}_{\hat{b}}\big(\Lambda(x(\xi)),\,k(\xi)\big)$ describes the LLT applied to each local laboratory at x, as the observer measures the local 4-momentum $k^{\hat{a}}(x)$ of the photon at x. Thus, the evolution of the photon helicity state in Eq.(50) is that measured by this class of observers (vs a single observer).

Suppose we have a bipartite photon helicity Bell state at the spacetime point x of the form

$$|\Phi(x)\rangle = |k_1^{\hat{a}}(x),\lambda_1\rangle\,|k_2^{\hat{b}}(x),\lambda_2\rangle \pm |k_1^{\hat{a}}(x),\lambda_2\rangle\,|k_2^{\hat{b}}(x),\lambda_1\rangle. \tag{51}$$

As a specific example, one might consider two photons of helicity λ_1 and λ_2, one traveling inward and the other traveling outward along a common geodesic (e.g. a $\pm\,k^r(x)$ in Eq.(38)).

The infinitesimal evolution of this state along the photon's trajectories is then

$$\begin{aligned} U(\Lambda)\,|\Phi(x)\rangle &= e^{i\lambda_1\,\psi(\Lambda,\vec{n}_1)}\,|k_1'^{\hat{a}}(x),\lambda_1\rangle\,e^{i\lambda_2\,\psi(\Lambda,\vec{n}_2)}\,|k_2'^{\hat{b}}(x),\lambda_2\rangle \\ &\pm\ e^{i\lambda_2\,\psi(\Lambda,\vec{n}_1)}\,|k_1'^{\hat{a}}(x),\lambda_2\rangle\,e^{i\lambda_1\,\psi(\Lambda,\vec{n}_2)}\,|k_2'^{\hat{b}}(x),\lambda_1\rangle, \\ &=\ |k_1'^{\hat{a}}(x),\lambda_1\rangle\,|k_2'^{\hat{b}}(x),\lambda_2\rangle \\ &\pm\ e^{-i(\lambda_1-\lambda_2)\,(\psi(\Lambda,\vec{n}_1)-\psi(\Lambda,\vec{n}_2))}\,|k_1'^{\hat{a}}(x),\lambda_2\rangle\,|k_2'^{\hat{b}}(x),\lambda_1\rangle, \end{aligned} \tag{52}$$

where in the last line we have dropped an overall phase. As we let these photons separate macroscopically we have to apply the time ordering operations as in Eq.(50)

$$\begin{aligned} U(\Lambda)\,|\Psi(x)\rangle \;=\; & T\,[e^{i\lambda_1\int\tilde{\psi}(\Lambda(\xi),\,\vec{n}_1(\xi))d\xi}]\,|\,T\,[e^{\int\tilde{\lambda}^{\hat{a}}{}_{\hat{c}}(\Lambda(\xi),\,k(\xi))d\xi}]\,k_1^{\hat{c}}(x),\lambda_1\rangle\;T\,[e^{i\lambda_2\int\tilde{\psi}(\Lambda(\xi),\,\vec{n}_2(\xi))d\xi}]\,|\,T\,[e^{\int\tilde{\lambda}^{\hat{b}}{}_{\hat{d}}(\Lambda(\xi),\,k(\xi))d\xi}]\,k_2^{\hat{d}}(x),\lambda_2\rangle \\ \pm\; & T\,[e^{i\lambda_2\int\tilde{\psi}(\Lambda(\xi),\,\vec{n}_1(\xi))d\xi}]\,|\,T\,[e^{\int\tilde{\lambda}^{\hat{a}}{}_{\hat{c}}(\Lambda(\xi),\,k(\xi))d\xi}]\,k_1^{\hat{c}}(x),\lambda_2\rangle\;T\,[e^{i\lambda_1\int\tilde{\psi}(\Lambda(\xi),\,\vec{n}_2(\xi))d\xi}]\,|\,T\,[e^{\int\tilde{\lambda}^{\hat{b}}{}_{\hat{d}}(\Lambda(\xi),\,k(\xi))d\xi}]\,k_2^{\hat{d}}(x),\lambda_1\rangle \end{aligned} \tag{53}$$

The lesson of the above expression is that over some macroscopic (non-infinitesimal) evolution of the photon trajectories, the relative phase between the product photon states (as in Eq.(52)) depends on the class of local

laboratories (tetrads) of the observers that the photons pass through. That is, we need to know the set of observers along the trajectory of the photons in order to determine the relative phase, since the state of motion of these observers (tetrads) determines the locally measured components of the direction of the photon $n_1(x(\xi))$ and $n_2(x(\xi))$, and hence the observer measured infinitesimal Wigner rotation angles $\tilde{\psi}(\Lambda(\xi),\,\vec{n}_j(\xi))$ at each spacetime point $x(\xi)$. For example, as discussed in Section IV B, for any motion of both the photon and observer in the equatorial plane, the phase factor in Eq.(52) is unity and the Bell state in Eq.(52) and Eq.(53) retains its original form of Eq.(51).

B. More General Photon States

In the above we have considered pure photon helicity states with a definite 4-momentum $\mathbf{k} \leftrightarrow k^{\hat{a}}$. In general, we can form wave packets states over a distribution of 4-momenta, and a linear combination of the helicities. The most general photon state has the form [15]

$$
\begin{aligned}
|\psi\rangle &= \sum_{\lambda=\pm1} \int \tilde{dk}\, \psi^{\lambda}(k)\, |\mathbf{k},\lambda\rangle, \qquad (54)\\
\tilde{dk} &= \frac{1}{(2\pi)3}\, \theta(k^{\hat{0}}) \delta^{(3)}(\vec{k}-\vec{k}')\, \delta^{\lambda'}_{\lambda} = \frac{d^3k}{(2\pi)^3 2k^{\hat{0}}},\\
&\sum_{\lambda=\pm1} \int \tilde{dk}\, |\psi^{\lambda}(k)|^2 = 1.\\
\langle \mathbf{k}',\lambda'|\mathbf{k},\lambda\rangle &= (2\pi)^3\, 2k^{\hat{0}}\, \delta^{(3)}(\vec{k}-\vec{k}')\, \delta^{\lambda'}{}_{\lambda} \equiv \tilde{\delta}^{(3)}(\vec{k}-\vec{k}')\, \delta^{\lambda'}{}_{\lambda},
\end{aligned}
$$

where $\tilde{dk}$ is the invariant Lorentz integration measure, and we have used the covariant normalization convention for the inner product of helicity states. A linear polarization state (LPS) of definite 4-momentum $\mathbf{k} = (k^{\hat{0}}, |\vec{k}|\,\vec{n})$ (where $\omega = |\vec{k}|$) and polarization angle ϕ is given by a linear combination of the two photon helicity states of the form

$$
|\mathbf{k},\phi\rangle \equiv |(k^{\hat{0}}, |\vec{k}|\,\vec{n}),\phi\rangle = \frac{1}{\sqrt{2}} \sum_{\lambda=\pm1} e^{i\lambda\phi}\, |\mathbf{k},\lambda\rangle. \qquad (55)
$$

In Eq.(55) the polarization angle ϕ is defined (as described in detail in the appendix) in the *standard* photon frame in which the photon propagates along the $\hat{\mathbf{z}}$-axis and the transverse polarization vectors lie in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane, one with angle ϕ with respect to the $\hat{\mathbf{x}}$-axis, and the other with angle $\phi+\pi/2$. The polarization angle ϕ is independent of the photon 4-momentum.

The most general LPS with definite direction $\vec{n}$ is given by the photon wave packet

$$
|g,\phi,\vec{n}\rangle = \frac{1}{\sqrt{2}} \sum_{\lambda=\pm1} e^{i\lambda\phi} \int d|\vec{k}|\, g(|\vec{k}|)\, |(k^{\hat{0}}, |\vec{k}|\,\vec{n}),\lambda\rangle. \qquad (56)
$$

We can form a reduced helicity density matrix of the state $|g,\phi,\vec{n}\rangle$ by projecting out the momenta

$$
\begin{aligned}
\rho_{red}(g,\phi,\vec{n}) &= \int \tilde{dk} \langle\mathbf{k}|g,\phi,\vec{n}\rangle\langle g,\phi,\vec{n}|\mathbf{k}\rangle\\
&= \sum_{\lambda\lambda'} \frac{1}{2} e^{i(\lambda-\lambda')\phi} |\lambda\rangle\langle\lambda'| = \frac{1}{2}\begin{pmatrix} 1 & e^{i2\phi}\\ e^{-i2\phi} & 1\end{pmatrix}, \qquad (57)
\end{aligned}
$$

where the rows and columns of the matrix are labeled by the helicity index in the order $\lambda = \{1,-1\}$.

In flat spacetime, Caban and Rembielinski [9] showed that under a SR LT this LPS remains a LPS, with a new polarization angle $\phi' = \phi + \psi(\Lambda,\vec{n})$ via

$$
\begin{aligned}
U(\Lambda)|g,\phi,\vec{n}\rangle &= |g',\phi,\vec{n}'\rangle, \quad g'(|\vec{k}|) = \frac{2}{a} g\left(\frac{2|\vec{k}|}{a}\right),\\
&= \frac{1}{\sqrt{2}} \sum_{\lambda=\pm1} e^{i\lambda(\phi+\psi(\Lambda,\vec{n}))} \int d|\vec{k}|\, g'(|\vec{k}|)\, |(k^{\hat{0}}, |\vec{k}|\,\vec{n}'),\lambda\rangle,
\end{aligned}
\qquad (58)
$$

where $\vec{n}$ and $g(|\vec{k}|)$ are the original photon propagation direction and wave packet momentum distribution, and $\vec{n}'$ and $g'(|\vec{k}|)$ are the corresponding transformed quantities. The authors showed that the reduced helicity density matrix for transformed state $U(\Lambda)|g,\phi,\vec{n}\rangle$ transforms properly under LTs, i.e.

$$
\begin{aligned}
\rho'_{red} &= U(\Lambda)\rho_{red}(g,\phi,\vec{n})U^{\dagger}(\Lambda)\\
&= \frac{1}{2}\begin{pmatrix} 1 & e^{i2(\phi+\psi(\Lambda,\vec{n}))}\\ e^{-i2(\phi+\psi(\Lambda,\vec{n}))} & 1\end{pmatrix}\\
&= \rho_{red}(g',\phi+\psi(\Lambda,\vec{n})),\vec{n}'), \quad U(\Lambda)_{\lambda\lambda'} = e^{i\lambda\psi(\Lambda,\vec{n})}\delta_{\lambda\lambda'}
\end{aligned}
\qquad (59)
$$

The fact that the LPS admits a covariant description of the reduced density matrix in terms of helicity degrees of freedom is related to the fact that LTs do not create entanglement between the helicity and momentum directions [9], as indicated by the matrix elements of $U(\Lambda)$ in the last line of Eq.(59) arising from fundamental transformation law Eq.(49). This is very different from the situation for massive particles in which the action of $U(\Lambda)$ Eq.(1) does entangle momentum and spin [6]. These considerations remain true in CST, where by the EP, SR holds locally at each spacetime point if we interpret $k^{\hat{a}}(x) = e^{\hat{a}}{}_{\alpha}(x)\, k^{\alpha}(x)$ as the components of the photon 4-momentum as measured by an observer at x with tetrad $\mathbf{e}_{\hat{a}}(x)$.

The transformation of $g \to g'$ in Eq.(58) arose from applying the unitary transformation Eq.(49) to the helicity states, making a change of integration variable from $|\vec{k}|$ to $|\vec{k}'|$ and using the frequency transformation law $k'^{\hat{0}} = ak^{\hat{0}}/2$ from Eq.(21) (with a final relabeling of $|\vec{k}'|$ to $|\vec{k}|$). Note that the transformed polarization angle $\phi' = \phi + \psi(\Lambda,\vec{n})$ involves the Wigner angle $\psi(\Lambda,\vec{n})$ evaluated at the original propagation direction of the photon

$\vec{n}$, while the kets on the right hand side are evaluated at the transformed photon direction $\vec{n}'$. Since $\psi(\Lambda, \vec{n})$ depends only on the direction of the photon and not on its frequency $\omega = |\vec{k}|$, the phase factor $e^{i\lambda\,\psi}$, resulting from the unitary transformation of the photon helicity states, can be pulled outside the integral.

A general two particle photon state takes the form

$$
\begin{aligned}
|\Psi\rangle &= \sum_{\lambda_1,\lambda_2} \int\int \tilde{d}k_1 \tilde{d}k_2 \, g_{\lambda_1\lambda_2}(\mathbf{k}_1, \mathbf{k}_2) |\mathbf{k}_1, \lambda_1\rangle \, |\mathbf{k}_2, \lambda_2\rangle, \\
& \sum_{\lambda_1,\lambda_2} \int\int \tilde{d}k_1 \tilde{d}k_2 \, |g_{\lambda_1\lambda_2}(\mathbf{k}_1, \mathbf{k}_2)|^2 = 1. \qquad (60)
\end{aligned}
$$

For the choice of the distribution function $g_{\lambda_1\lambda_2}(\mathbf{k}_1, \mathbf{k}_2)$

$$
g_{\lambda_1\lambda_2}(\mathbf{k}_1, \mathbf{k}_2) = \frac{1}{\sqrt{2}} \, e^{i\lambda_1\phi_1} \, e^{i\lambda_2\phi_2} \, \delta_{\lambda_1\lambda_2} f(\mathbf{k_1}, \mathbf{k_2}), \qquad (61)
$$

where ϕ_1 and ϕ_2 constant, momentum independent polarization angles, the state $|\Psi\rangle$ is fully entangled in helicity and entangled in momentum if f is non-factorizable, i.e. $f(\mathbf{k_1}, \mathbf{k_2}) \neq f_1(\mathbf{k}_1)\, f_2(\mathbf{k}_2)$. In analogy with the single particle state Eq.(57), the helicity reduced density matrix obtained by tracing the pair of momenta is

$$
\rho_{red}(\phi_1, \phi_2) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & e^{i2(\phi_1+\phi_2)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-i2(\phi_1+\phi_2)} & 0 & 0 & 1 \end{pmatrix}, \qquad (62)
$$

where the normalization condition for $g_{\lambda_1\lambda_2}$ in Eq.(60) has been used, and the rows and columns of the matrix are labeled by the double helicity indices $\lambda_1\lambda_2 = \{11,\, 1-1,\, -11,\, -1-1\}$. Under the action of an infinitesimal LLT $\mathcal{U}(\Lambda) = U(\Lambda) \otimes U(\Lambda)$ the reduced helicity density matrix for the state $\mathcal{U}(\Lambda)\,|\Psi\rangle$ is

$$
\begin{aligned}
\rho'_{red}(\phi_1, \phi_2) &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & (\rho'_{red})_{11,-1-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (\rho'_{red})^*_{11,-1-1} & 0 & 0 & 1 \end{pmatrix} \\
(\rho'_{red})_{11,-1-1} &= \int\int \tilde{d}k_1 \tilde{d}k_2 |f'(\mathbf{k}_1, \mathbf{k}_2)|^2 e^{i2\Phi(\vec{n}_1, \vec{n}_2)} \\
\Phi(\vec{n}_1, \vec{n}_2) &= \phi_1 + \psi(\Lambda, \vec{n}_1) + \phi_2 + \psi(\Lambda, \vec{n}_2). \qquad (63)
\end{aligned}
$$

In Eq.(63) f' is the transformed distribution function (analogous to g' in Eq.(58)) that depends on the untransformed frequencies $|\vec{k}_j|$, but the transformed directions $\vec{n}'_j$. Since $\tilde{d}k_j \propto d^3k_j = \delta\Omega_{\vec{n}_j}\, d|\vec{k}_j|\, |\vec{k}_j|^2$ involves an integration over the untransformed photon directions $\vec{n}_j$, (where $\vec{n}'_j$ and $\vec{n}_j$ are related by Eq.(21)), the integral over $\delta\Omega_{\vec{n}_j}$ is in general very complicated, mixing up the photon directions, but again without entangling with the helicity. Without the factors of $\psi_{\vec{n}_j}$ in the argument of the phase of $(\rho'_{red})^*_{11,-1-1}$ Eq.(63) reduces to Eq.(62), showing that analogous to Eq.(59), the reduced helicity density matrix $\rho'_{red}(\phi_1, \phi_2)$ transforms covariantly under LLTs. In CST, it is possible that the observer (tetrad) changes for each LLT along the trajectory of the photons, from which the local Wigner angles $\psi_{\vec{n}_i}(x(\xi))$ are measured by (massive) observers.

VII. SUMMARY AND CONCLUSIONS

The Wigner rotation for a photon can be envisioned as the rotation of the transverse linear polarization vectors, in the plane perpendicular to the direction of propagation of the photon, resulting from a Lorentz transformation Λ between observers. The natural quantum state description of the photon is in terms of helicity states $|\mathbf{k}, \lambda\rangle$, $\lambda = \pm 1$, $\mathbf{k} = (k^{\hat{0}}, |\vec{k}|\vec{n})$, in which the corresponding induced unitary transformation $U(\Lambda)$ introduces a phase factor, dependent upon Λ and the propagation direction of the photon $\vec{n}$, without changing the helicity, i.e. $U(\Lambda)|\mathbf{k}, \lambda\rangle = e^{i\lambda\psi(\Lambda,\vec{n})}|\mathbf{k'}, \lambda\rangle$. In the flat spacetime of special relativity Λ transforms between a special class of observers, namely inertial observers for which the acceleration of the observer is zero (constant velocity observers). Such observers are global in the sense that they are position independent and exist over the whole of the flat spacetime.

In going to curved spacetime (CST) where general relativity applies, all types of observers, in arbitrary states of motion, are allowed. The motion of these observers is now reduced to a local description, encapsulated in an orthonormal tetrad $\mathbf{e}_{\hat{a}}(x)$ that describes the four axes (three spatial and one temporal) that defines the observer's local laboratory at the spacetime point x from which he makes measurements. For example, the photon 4-momentum $k^\alpha(x)$ existing in a CST described by coordinates x^α, has components $k^{\hat{a}}(x)$ in the observer's local laboratory given by $k^{\hat{a}}(x) = e^{\hat{a}}{}_\alpha(x)\, k^\alpha(x)$.

By the equivalence principle, the laws of special relativity apply in this local laboratory (local tangent plane to the curved spacetime), at the spacetime point x. Therefore, we can compute the Wigner rotation angle in CST by applying the calculational procedure appropriate for flat spacetime to the observer's instantaneous local laboratory. The quantum state of the photon is described by the local helicity state $|k^{\hat{a}}(x), \lambda\rangle$ and the observer by the tetrad $\mathbf{e}_{\hat{a}}(x)$. The instantaneous Wigner rotation angle $\psi(\Lambda, \vec{n})$, arising from a local Lorentz transformation as the photon traverses infinitesimally along its geodesic, now depends on the propagation direction of the photon $(\vec{n})^{\hat{i}} = k^{\hat{i}}/|\vec{k}|$ as measured locally by the observer at x.

In this work we have developed the local Wigner rotation for photons in an arbitrary CST. We have given specific examples in the case of Schwarzschild spacetime and compared these with the results from flat spacetime. The difference in the CST case is that an explicit description of the observer, via his tetrad, is needed to compute the local Wigner angle. That is, the locally measured Wigner rotation angle is observer dependent, which we develop explicitly. In terms of a local helicity state de-

scription of the quantum photon states, the induced local Lorentz transformation that gives rise to the local Wigner angle as the photon traverses its geodesic, does not entangle the photon direction with the helicity, in contrast to the case for spin-momentum entanglement that occurs for massive particles. We have also developed a sufficient condition for observers who would measure zero Wigner rotation and have shown that such observers can be in Fermi-Walker frames, i.e. the instantaneous non-rotating rest frame of the accelerating observer.

Acknowledgments

PMA wishes to acknowledge the support of the Air Force Office of Scientific Research (AFOSR) for this work.

APPENDIX A: WIGNER ROTATION IN FLAT SPACETIME: EXAMPLES

In this appendix we give explicit examples illustrating the operational meaning of the Wigner rotation in flat spacetime in terms of its effect on the polarization vectors for photons.

As given in Eq.(6), the polarization vectors for positive and negative helicity states $\epsilon^\mu_\pm(\hat{\boldsymbol{k}})$ (right and left circular polarization) with propagation 4-vector $\mathbf{k} = (k^0, |\vec{k}|\,\vec{n})$ are given by

$$\epsilon^\mu_\pm(\hat{\boldsymbol{k}}) = \frac{R(\hat{\boldsymbol{k}})}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ \mp i \\ 0 \end{bmatrix}, \tag{A1}$$

with the components of the column vector labeled by the Cartesian coordinates $x^\mu = (t, x, y, z)$. Here $R(\hat{\boldsymbol{k}})$ is the rotation that takes the *standard* direction $\hat{\mathbf{z}}$-axis to the photon propagation direction $\hat{\boldsymbol{k}} = \vec{k}/|\vec{k}|\,(= \vec{n})$. Under a LT Λ, the polarization vector transforms as $\epsilon^\mu_\pm \to \epsilon'^{\mu}$ with [8]

$$\begin{aligned} \epsilon'^{\mu}_\pm(\hat{\boldsymbol{k}}') &\equiv D(\Lambda)\,\epsilon^\mu_\pm(\hat{\boldsymbol{k}}) \\ &= R(\Lambda\hat{\boldsymbol{k}})\,R_z(\psi(\Lambda,\vec{n}))\,R(\hat{\boldsymbol{k}})^{-1}\,\epsilon^\mu_\pm(\hat{\boldsymbol{k}}), \qquad \text{(A2)} \\ &= \Lambda\epsilon^\mu_\pm(\hat{\boldsymbol{k}}) - \frac{(\Lambda\,\epsilon^\mu_\pm(\hat{\boldsymbol{k}}))^0}{(\Lambda\,k^\mu)^0}\,\Lambda k^\mu. \qquad \text{(A3)} \end{aligned}$$

Here we use the typical "abuse of notation" denoting $\Lambda\,\hat{\boldsymbol{k}}$ for the transformed photon direction $\hat{\boldsymbol{k}}' = \vec{k}'/|\vec{k}'|$, where $\vec{k}'$ is the 3-vector portion of the transformed photon 4-momentum $k'^{\mu} = \Lambda^\mu{}_\nu\,k^\nu$. Thus $R(\Lambda\,\hat{\boldsymbol{k}})$ is the rotation taking the standard direction $\hat{\mathbf{z}}$ to $\hat{\boldsymbol{k}}'$.

From Eq.(A1) we can construct a linear polarization vector (LPV) $\epsilon^\mu_\phi(\hat{\boldsymbol{k}})$

$$\begin{aligned} \epsilon^\mu_\phi(\hat{\boldsymbol{k}}) &= \frac{1}{\sqrt{2}}\left(e^{i\phi}\,\epsilon^\mu_+(\hat{\boldsymbol{k}}) + e^{-i\phi}\,\epsilon^\mu_-(\hat{\boldsymbol{k}})\right), \\ &= R(\hat{\boldsymbol{k}})\begin{bmatrix} 0 \\ \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} \equiv R(\hat{\boldsymbol{k}})\,\tilde{\epsilon}^\mu_\phi(\hat{\mathbf{z}}), \qquad \text{(A4)} \end{aligned}$$

where the polarization angle ϕ is defined by the angle the LPV $\tilde{\epsilon}^\mu_\phi(\hat{\mathbf{z}})$ makes with the $\hat{\mathbf{x}}$-axis when the photon propagates along the $\hat{\mathbf{z}}$-axis in this *standard frame*, i.e. $\hat{\tilde{\mathbf{k}}} = \hat{\mathbf{z}}$ (denoted by *tildes* over vectors). The LPV in the standard frame $\tilde{\epsilon}^\mu_\phi(\hat{\mathbf{z}})$ is obtained by rotating the polarization vector $\epsilon^\mu_\phi(\hat{\boldsymbol{k}})$ propagating in the direction $\hat{\boldsymbol{k}}$, by the transformation that takes $\hat{\boldsymbol{k}}$ back to $\hat{\mathbf{z}}$, i.e. by the rotation $R^{-1}(\hat{\boldsymbol{k}})$.

After a LT Λ, the new LPV is given by

$$\begin{aligned} \epsilon'^\mu_{\phi'}(\hat{\boldsymbol{k}}') &= \frac{1}{\sqrt{2}}\left(e^{i\phi'}\,\epsilon'^\mu_+(\hat{\boldsymbol{k}}') + e^{-i\phi'}\,\epsilon'^\mu_-(\hat{\boldsymbol{k}}')\right), \\ &= R(\Lambda\,\hat{\boldsymbol{k}})\begin{bmatrix} 0 \\ \cos\phi' \\ \sin\phi' \\ 0 \end{bmatrix} \equiv R(\Lambda\,\hat{\boldsymbol{k}})\,\tilde{\epsilon}'^\mu_{\phi'}(\hat{\mathbf{z}}). \qquad \text{(A5)} \end{aligned}$$

Again, the transformed LPV $\tilde{\epsilon}'^\mu_{\phi'}(\hat{\mathbf{z}})$ in the standard frame ($\hat{\tilde{\mathbf{k}}}' = \hat{\mathbf{z}}$) is obtained by rotating the polarization vector $\epsilon'^\mu_{\phi'}(\Lambda\,\hat{\boldsymbol{k}})$ propagating in the direction $\hat{\boldsymbol{k}}'$, by the rotation that takes $\hat{\boldsymbol{k}}'$ back to $\hat{\mathbf{z}}$, i.e. $R^{-1}(\Lambda\,\hat{\boldsymbol{k}})$. The angle that $\tilde{\epsilon}'^\mu_{\phi'}(\hat{\mathbf{z}})$ makes with the $\hat{\mathbf{x}}$-axis in the standard frame defines the transformed polarization angle ϕ'.

By multiplying Eq.(A2) by $R^{-1}(\Lambda\,\hat{\boldsymbol{k}})$ and inserting Eq.(A4) and Eq.(A5) we obtain

$$\tilde{\epsilon}'^\mu_{\phi'}(\hat{\mathbf{z}}) = R_z(\psi(\Lambda,\vec{n}))\,\tilde{\epsilon}^\mu_\phi(\hat{\mathbf{z}}), \tag{A6}$$

which upon comparing the arguments of the trigonometric functions yields

$$\phi' = \phi + \psi(\Lambda,\vec{n}). \tag{A7}$$

This states that the effect of a LT Λ is a rotation of the polarization angle $\phi \to \phi'$ in the standard frame by the Wigner angle $\psi(\Lambda,\vec{n})$, i.e. a rotation of the *standard* polarization vectors once we bring them back to the *standard frame* in which the photon propagates along the $\hat{\mathbf{z}}$-axis, ($\hat{\tilde{\mathbf{k}}} = \hat{\mathbf{z}}$), and the standard polarization vectors lie in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane. It is in this standard frame that we most easily measure the polarization angles ϕ and ϕ' and therefore determine the Wigner angle $\psi(\Lambda,\vec{n})$. In the following we illustrate a few sample cases in which the Wigner angle is zero, and non-zero.

In Fig.(2) we consider the photon to be traveling along the $\hat{\mathbf{z}}$-axis, and consider a boost along the $\hat{\mathbf{x}}$-axis. We denote the boost direction by $\hat{\mathbf{e}} \equiv \vec{v}/|\vec{v}|$ where $\vec{v}$ is the

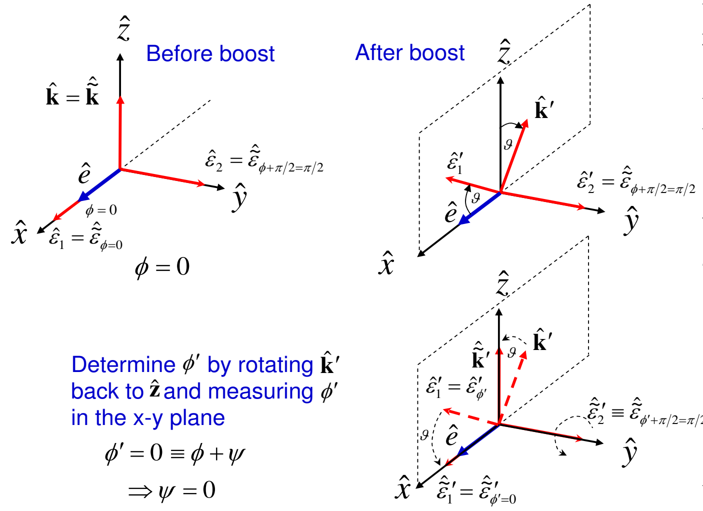

FIG. 2: (Color online) Example of a zero Wigner angle ψ: photon propagation direction $\hat{\boldsymbol{k}}$ along the $\mathbf{z}$-axis, boost $\hat{\mathbf{e}}$ along the $\mathbf{z}$-axis. In fact, the Wigner angle ψ is identically zero for any boost direction $\hat{\mathbf{e}}$ if the photon $\hat{\boldsymbol{k}}$ travels along the $\mathbf{z}$-axis.

velocity of the frame we are transforming to and ξ defined by $\tanh\xi = -|\vec{v}|/c$ is the rapidity parameter of the boost. We take the two transverse polarization vectors to lie along the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ axes, so that the $\hat{\mathbf{x}}$ polarization vector has a polarization angle $\phi = 0$. The boost rotates the propagation vector $\hat{\boldsymbol{k}}$ and the $\hat{\mathbf{x}}$ polarization vector in the $\hat{\mathbf{x}}$-$\hat{\mathbf{z}}$ plane counterclockwise about the $\hat{\mathbf{y}}$-axis by some boost dependent angle ϑ [5, 8], leaving the $\hat{\mathbf{y}}$ polarization vector unchanged. To determine ϕ' we rotate $\hat{\boldsymbol{k}}'$ back to the $\hat{\mathbf{z}}$-axis clockwise about the $\hat{\mathbf{y}}$-axis, undoing the original rotation, and thus returning all the vectors to their original orientations. Therefore, $\phi' = \phi$ and thus the Wigner angle is zero. It is straightforward to show that for the photon traveling along the $\hat{\mathbf{z}}$-axis, a boost along *any* direction yields a zero Wigner angle.

In general for $\hat{\boldsymbol{k}}$ not along $\hat{\mathbf{z}}$, we can determine the polarization angles ϕ and ϕ' as follows. Given a photon propagating along the direction $\hat{\boldsymbol{k}}$ with polarization vectors in a plane perpendicular to this vector, we find the polarizations vectors in the standard frame by applying the rotation $R_{\hat{\boldsymbol{k}}\times\hat{\mathbf{z}}}(\theta)$ to the triad, where $\cos\theta = \hat{k}_3$ which takes $\hat{\boldsymbol{k}} \to \hat{\mathbf{z}}$. A pure boost along the $\hat{\mathbf{e}}$ rotates the photon propagation direction by a boost dependent angle ϑ counterclockwise about the axis $\hat{\mathbf{e}}\times\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{k}}' = R_{\hat{\mathbf{e}}\times\hat{\boldsymbol{k}}}(\vartheta)\,\hat{\boldsymbol{k}}'$. To determine the transformed polarization angle ϕ' we rotate $\hat{\boldsymbol{k}}'$ counter clockwise along the direction $\hat{\boldsymbol{k}}'\times\hat{\mathbf{z}}$ by the angle θ' where $\cos\theta' = \hat{k}'_3$, which takes $\hat{\boldsymbol{k}}' \to \hat{\mathbf{z}}$, and the transformed polarization vectors to the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane.

From the above discussion we can also infer that the Wigner angle is zero if $\hat{\mathbf{e}}$ and $\hat{\boldsymbol{k}}$ both lie in the $\hat{\mathbf{x}}$-$\hat{\mathbf{z}}$ plane, or both lie in the $\hat{\mathbf{y}}$-$\hat{\mathbf{z}}$ plane. In the latter case, if the first polarization vector also lies in the $\hat{\mathbf{y}}$-$\hat{\mathbf{z}}$ plane orthogonal to $\hat{\boldsymbol{k}}$, and the second polarization vector lies along the $\hat{\mathbf{x}}$-axis, the a boost in the $\hat{\mathbf{y}}$-$\hat{\mathbf{z}}$ plane will rotate the triad $(\hat{\boldsymbol{k}}, \hat{\boldsymbol{\epsilon}}_1, \hat{\boldsymbol{\epsilon}}_2{=}\hat{\mathbf{x}})$ about the $\hat{\mathbf{x}}$-axis. Therefore, when we undo this rotation about the $\hat{\mathbf{x}}$-axis in order to calculate ϕ', the triad is returned to its original orientation, so that $\phi' = \phi$, implying a zero Wigner angle, analogous to Fig.(2).

However, if $\hat{\mathbf{e}}$ and $\hat{\boldsymbol{k}}$ both lie in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane, the Wigner angle is non-zero, as illustrated in Fig.(3) for the case of $(\hat{\boldsymbol{k}}{=}\hat{\mathbf{x}}, \hat{\boldsymbol{\epsilon}}_1{=}\hat{\mathbf{y}}, \hat{\boldsymbol{\epsilon}}_2{=}\hat{\mathbf{z}})$. Here the initial polarization angle for $\hat{\boldsymbol{\epsilon}}_1$ is $\phi = \pi/2$. For a boost in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane, as indicated in the figure, $\hat{\boldsymbol{k}}$ is rotated counter clockwise about the $\hat{\mathbf{z}}$-axis by the boost dependent angle ϑ, yielding a transformed polarization vector $\hat{\boldsymbol{\epsilon}}'_1$ oriented at angle $\phi' = \pi/2 - \vartheta$ with respect to the $\hat{\boldsymbol{x}}$-axis. Upon rotating $\hat{\boldsymbol{k}}'$ (about $\hat{\boldsymbol{\epsilon}}'_1$) to the $\hat{\mathbf{z}}$-axis, $\hat{\boldsymbol{\epsilon}}'_1$ is left invariant, maintaining the relation $\phi' = \pi/2 - \vartheta$, or equivalent, a Wigner angle of $\psi(\Lambda, \vec{n}) = -\vartheta$.

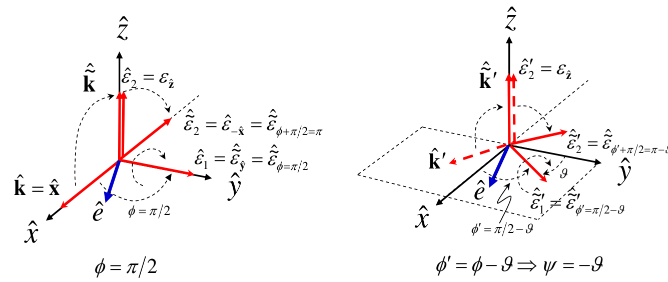

FIG. 3: (Color online) Example of a non-zero Wigner angle ψ: photon propagation direction $\hat{\boldsymbol{k}}$ along the $\mathbf{x}$-axis, boost $\hat{\mathbf{e}}$ in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane.

In Fig.(4) we illustrate one last case that is relevant to the discussion in the main body of the text when we consider an example of a non-zero Wigner angle in curved Schwarzschild spacetime. Here in flat spacetime, we consider the case when the photon direction $\hat{\boldsymbol{k}}$ is in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane at some angle φ to the $\hat{\mathbf{x}}$-axis, and the boost direction $\hat{\mathbf{e}}$ is in the $\hat{\mathbf{y}}$-$\hat{\mathbf{z}}$ plane at some polar angle θ with respect to the $\hat{\mathbf{z}}$-axis. We choose the first polarization vector to lie along the $\hat{\mathbf{z}}$-axis, and the second polarization vector to lie in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ orthogonal to $\hat{\boldsymbol{k}}$ with polarization angle $\phi = \varphi - \pi/2$, A boost along $\hat{\mathbf{e}}$ induces a rotation of the triad $(\hat{\boldsymbol{k}}, \hat{\boldsymbol{\epsilon}}_1 = \hat{\mathbf{z}}, \hat{\boldsymbol{\epsilon}}_2)$ about the axis $(\hat{\mathbf{e}}\times\hat{\boldsymbol{k}}$ orthogonal to the $\hat{\boldsymbol{k}}$-$\hat{\mathbf{e}}$ plane by some boost dependent angle ϑ. This pushes $\hat{\boldsymbol{k}}'$ below the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane and pulls $\hat{\boldsymbol{\epsilon}}'_2$ above the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane. Upon rotating $\hat{\boldsymbol{k}}'$ back to the $\hat{\mathbf{z}}$-axis by a rotation about the direction $\hat{\boldsymbol{k}}'\times\hat{\mathbf{z}}$, $\hat{\boldsymbol{\epsilon}}'_2$ is returned to the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane as $\hat{\tilde{\boldsymbol{\epsilon}}}'_2$. For a boost of infinitesimal angle $\delta\varphi$ as illustrated in Fig.(4), the polarization angle of the transformed polarization vector is $\phi' = \phi + \psi(\Lambda, \vec{n})$ with the non-zero Wigner angle given to $O(\delta\vartheta)$ as $\psi(\Lambda, \vec{n}) = \delta\vartheta\sin\theta\cos\varphi = \delta\vartheta\,\hat{k}_3$.

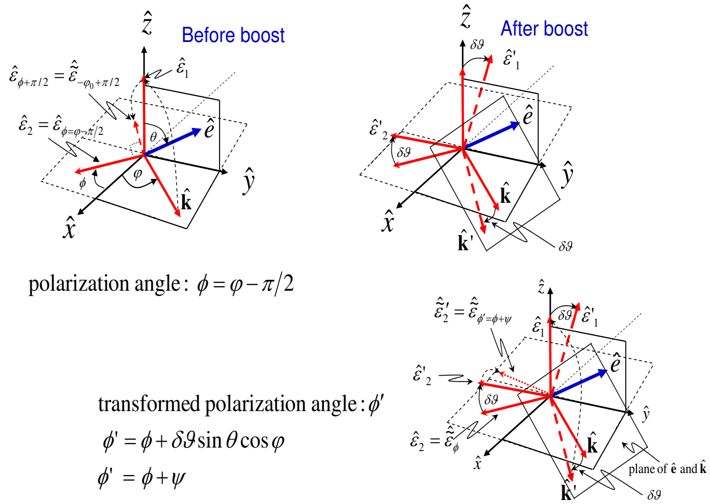

FIG. 4: (Color online) Second example of non-zero Wigner angle ψ: photon propagation direction $\hat{\boldsymbol{k}}$ in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane at azimuthal angle φ, boost $\hat{\mathbf{e}}$ in the $\hat{\mathbf{y}}$-$\hat{\mathbf{z}}$ plane at polar angle θ.

[1] A. Peres and D.R. Terno, Rev. Mod. Phys. **76**, 93 (2004).

[2] S. Weinberg, *Quantum Theory of Fields: Vol-I*, Cambridge Univ. Press, Cambridge, 62-74 (1995).

[3] P.M. Alsing, G.J. Stephenson Jr. and P. Kilian *Spin-induced non-geodesic motion, gyroscopic precession, Wigner rotation and EPR correlations of massive spin-$\frac{1}{2}$ particles in a gravitational field*, submitted to Phys. Rev. A, February (2009); arXiv:0902.1396 [quant-ph].

[4] A. Peres, P.F. Scudo and D.R. Terno, Phys. Rev. Lett **88**, 230402 (2002).

[5] P.M. Alsing and G.J. Milburn, Quantum Inf. Comput. **2**, 487 (2002).

[6] R.M. Gingrich and C. Adami, Phys. Rev. Lett. **89**, 270402 (2002).

[7] W.K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[8] R.M. Gingrich, A.J. Bergou and C. Adami, quant-ph/0302095.

[9] P. Caban and J. Rembielinski, Phys. Rev. A **68**, 042107 (2003).

[10] H. Terashima and M. Ueda, Int. J. Quant. Info. 1, 93 (2003). *ibid*, Phys. Rev. A **69** 032113 (2004); quant-ph/0307114.

[11] See, for example, C.W. Misner, K.S. Thorne and J.A. Wheeler (MTW), *Gravitation*, W.H. Freeman and Co., San Francisco, CA, 207-216, 327-332 (1973); S.M. Carroll, *Spacetime and Geometry*, Addison Wesley, San Francisco, CA, 483-494 (2004). $\nabla_\beta\, k_\mu = \partial_\beta\, k_\mu - \Gamma^\lambda{}_{\alpha\beta}\, k_\lambda$ where the Christoffel connection $\Gamma^\lambda{}_{\alpha\beta}(x)$ is obtained from the metric as $\Gamma^\lambda{}_{\alpha\beta} = \frac{1}{2}\, g^{\lambda\sigma}\, (\partial_\alpha\, g_{\beta\lambda} + \partial_\beta\, g_{\alpha\lambda} - \partial_\lambda\, g_{\alpha\beta})$.

[12] The force free (zero acceleration) equations of motion of a (massive or massless) particle can be obtained from the Euler-Lagrange equations $d/d\sigma(\partial L/\partial\dot{x}^\alpha) - \partial L/\partial x^\alpha = 0$, where $L = ds = (g_{\alpha\beta}(x)\, \dot{x}^\alpha\, \dot{x}^\beta)^{1/2}$ is the effective Lagrangian, and $\dot{x}^a = dx^\alpha/d\sigma$ with σ the affine paramter along the particle's geodesic trajectory. If the metric is independent of the coordinate $x_0^\alpha \equiv x_o$, the quantity $\partial L/\partial\dot{x}_o$ is a constant of the motion along the geodesic. The latter expression can be written as $\boldsymbol{\zeta}\cdot\mathbf{u} =$ constant where $u^\alpha = \dot{x}^\alpha$ is the 4-velocity of the particle, i.e. the tangent to its geodesicc, and $\zeta^\alpha = \delta^\alpha{}_{x_o}$ is called a *Killing vector*, and reprsents an isometry of the metric. The Schwarzschild metric Eq.(28) is independent of the coordinates t and ϕ. The quantity $e = \boldsymbol{\zeta}\cdot\mathbf{u} = (1 - r_s/r)\,\dot{t}$ for $\zeta^\alpha = \delta^\alpha{}_t$ represents the particle's energy (per unit mass for $m \neq 0$), and $l = \boldsymbol{\zeta}\cdot\mathbf{u} = r^2 \sin^2\theta\,\dot{\phi}$ for $\zeta^\alpha = \delta^\alpha{}_\phi$ represents the particle's orbital angular momentum (per unit mass for $m \neq 0$).

[13] James B. Hartle, *Gravity*, Addison-Wesley, San Francisco, CA, 193-204, 298-302, 440-441 (2003).

[14] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields, 4th. Ed.*, Pergamon Press, N.Y., 291-294 (1975); J. Evans, K.K. Nandi and A. Islam, Am. J. Phys. **64**, 1404, (1996), P.M. Alsing, Am. J. Phys. **66**, 779 (1998), J. Evans, P.M. Alsing, S. Giorgetti and K.K. Nandi, Am. J. Phys. **69**, 1103 (2001), P.M. Alsing, J.C. Evans and K.K. Nandi, Gen. Rel. and Grav. **33**, 1459-1487 (2001); gr-qc/0010065.

[15] Tung, Wu-Ki, *Group Theory in Physics*, World Scientific, Philadelphia, PA (1985).